Welfare, Population Growth and Dynamic Inefficiency in an OLG Framework

Daniel Mulino *
Department of Economics, Monash University

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Abstract

In a two-period OLG framework, there is only one rate of population growth at which the competitive equilbrium outcome is also the golden rule outcome. I show that this is the welfare *minimizing* outcome for agents. Moreover, I show that as population growth increases beyond the welfare minimizing level, agents are better off even as the economy becomes more dynamically inefficient.

1 Introduction

One of the key differences between the OLG and infinitely lived agent frameworks is the link between demographics and welfare. Samuelson (1975, 1976) and Deardorff (1976) compared the welfare of agents in a two period OLG framework at different rates of population growth when a social planner ensures that the economy has the golden rule level of capital. They showed that, under certain conditions, there is a welfare minimizing rate of population growth. Agents are increasingly well off as population growth diverges further away from this point - either higher or lower. This note generalizes the earlier result and extends the analysis to competitive equilibrium outcomes.

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I show that there is only one rate of population growth at which the competitive equilbrium outcome is also the golden rule outcome - and that this is the welfare *minimizing* outcome for agents. Moreover, I show that when population growth is higher than this welfare-minimizing level, the economy is dynamically inefficient yet agents are better off. As population growth increases, the economy becomes more dynamically inefficient and agents are increasingly better off.

2 Demographics and Welfare with a Social Planner

First, I examine the impact of demographics in an economy with a social planner. Where there is a social planner, an economy will attain the golden rule capital stock associated with each rate of population growth, although there may be a transition path at low or negative levels of population growth since large capital stocks need to be accumulated to reach the optimal steady state capital/labor ratio. At the golden rule level of capital per worker, the social planner may have to implement transfer payments in order to maximize the steady state utility of the representative agent.

The production function is CRS with two inputs, labor and capital: $Y_t = F(K_t, L_t)$. Each person inelastically supplies one unit of labor in the first period of their lives and none in the second period. The population grows at a constant rate, such that: $L_t = (1+n)L_{t-1}$. Both factors are paid their marginal product. Capital depreciates at rate δ. Therefore, the feasibility constraint for each period t is:

$$C_{1,t} + C_{2,t-1} + K_{t+1} = F(K_t, L_t) + (1-\delta)K_t$$

where $C_{1,t}$ denotes the total consumption of young agents born in period t and $C_{2,t-1}$ denotes the total consumption of old agents born in period $t-1$. This can be expressed in per capita terms:

$$c_{1,t} + \frac{c_{2,t-1}}{1+n} + (1+n)k_{t+1} = f(k_t) + (1-\delta)k_t$$

where $c_{1,t}$ denotes the consumption while young of agents born in period t, $c_{2,t}$ denotes consumption while old of agents born in period t and k_t denotes the capital per worker held by agents born in period t (ie who will be old agents when it is used in period $t+1$). In steady-state, we have:

$$c_1 + \frac{c_2}{1+n} + (n+\delta)k = f(k)$$

A social planner will solve a two part problem. First, the social planner will seek to maximize $f(k) - (n+\delta)k \;= c_1 + \frac{c_2}{1+n}$, which will be determined by the capital/labor ratio. Given the feasibility contraint, the social planner will maximize total steady-state consumption by choosing the optimal level of k. This will be achieved with the following first order condition: $f\prime(k) = n + \delta$. The social planner will then have to determine how to allocate consumption between generations. It may be necessary to implement transfer payments to achieve the optimal outcome. If each agent's lifetime utility is defined as: $U_t = u(c_{1,t}) + \beta u(c_{2,t})$, then utility will be maximized when the following condition holds:

$$u'(c_2)(1+n)\beta = u'(c_1)$$

2.1 Example: Cobb-Douglas Production; Log Utility

The first order condition that defines the optimal capital/labor ratio can be used to define the per capita level of capital holding at each rate of population growth:

$$\alpha k^{\alpha-1} = n + \delta \rightarrow k = \left(\frac{n+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

The social planner will also need to allocate output such that:

$$\frac{1}{c_2}(1+n)\beta = \frac{1}{c_1}$$

This implies the following consumption path:

$$c_1 = (1-\alpha)\left(\frac{n+\delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \frac{1}{1+\beta}$$

$$c_2 = (1+n)(1-\alpha)\left(\frac{n+\delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \frac{\beta}{1+\beta}$$

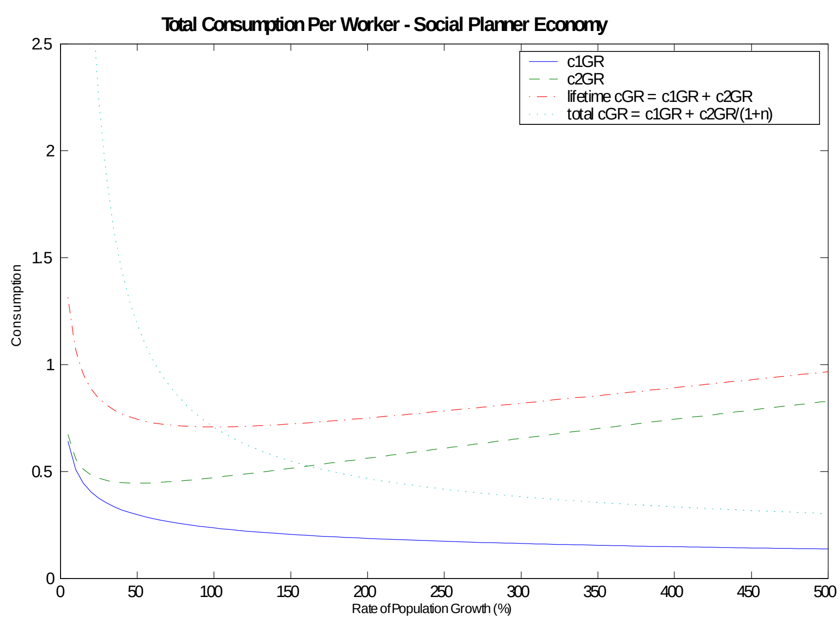
With Cobb-Douglas production and log utility, lifetime utility is u-shaped in relation to population growth. Lifetime utility will be minimized when:

$$n = \frac{\alpha(1+\beta) - \beta\delta(1-\alpha)}{\beta - \alpha - 2\alpha\beta}$$

Where $\beta = 1$, $\alpha = 1/4$ and $\delta = 0.5$, the welfare minimizing rate of population growth is 50 percent, which is an annual rate of 1.3 percent if one period is considered to be equivalent to 30 years. Where β is low, the welfare minimizing rate of population growth can be negative.

Capital Shallowing In both the OLG and infinitely lived agent models, as n falls, the optimal capital stock per worker rises. In turn, $c_1 + \frac{c_2}{1+n}$ rises as n falls. This can be seen from figure 1, which plots the optimal c_1 (thin solid line) and c_2 (thin dashed line) and the total output per worker available for consumption at each rate of population growth for the example outlined above.

Figure 1: Total Consumption Per Worker: C-D Production, Log Utility



The dotted line shows the relationship between n and $c_1 + \frac{c_2}{1+n}$. It is decreasing for all n. In contrast, the dash-dot line representing the relationship between n and $c_1 + c_2$ falls initially and then rises. The scale of Figure 1 incorporates very high levels of population growth to highlight this contrast. The rise in $c_1 + c_2$ reflects two offsetting forces. A higher n will reduce output per worker due to capital shallowing. However, a higher n will also reduce the number of dependents per worker. This will make it feasible for the consumption of each old person to rise by more than the consumption of each worker person falls even as output per worker falls. This is not only feasible, but also welfare-maximizing given the optimal allocation of output between periods. It is the increasing lifetime consumption stream of agents, even as the economy experiences capital shallowing, that drives the u-shaped welfare curve.

3 Competitive Market Economy

In the two period OLG framework, agents' welfare in the competitive equilibrium will either be monotonically decreasing as population growth increases (as with the infinitely lived agent result) or, if the intertemporal elasticity of substitution is high enough, it will be u-shaped. I show this for the Cobb-Douglas production log utility case and then generalize the result, demonstrating that : (i) welfare is minimized at the only rate of population growth where the equilibrium outcome is also the golden rule outcome; and (ii) agents' welfare is higher where faster population growth exacerbates an economy's dynamic inefficiency.

3.1 Case 1: Cobb-Douglas Production; Log Utility

The feasibility conditions are the same as in the social planner model. Agents divide the wage that is earned between first period consumption and savings. Second period consumption is equal to first period savings and the return earned on those savings. Agents maximize lifetime utility: $U_t = \log(c_{1,t}) + \beta \log(c_{2,t})$, subject to the following lifetime budget constraint:

$$c_{1,t} + \frac{c_{2,t}}{1 + r_{t+1}} = w_t$$

Clearly, with log utility, the saving rate will be independent of the interest rate: $s = \frac{\beta}{1+\beta}$. Since $K_t = sw_{t-1}L_{t-1}$, this implies a capital labor ratio of:

$$k_t = \frac{L_{t-1}sw_{t-1}}{L_t} = \frac{sw_{t-1}}{1+n} = \frac{\beta w_{t-1}}{(1+n)(1+\beta)}$$

Since $w_t = (1-\alpha)(k_t)^\alpha$, we can solve for the steady-state k:

$$k = \frac{\beta(1-\alpha)(k)^\alpha}{(1+n)(1+\beta)} \rightarrow k = \left[\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}$$

k is decreasing in n, which implies that r is increasing in n. The interest rate will be equal to:

$$r = \alpha\left[\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{\alpha-1}{1-\alpha}} - \delta = \frac{(1+n)(1+\beta)\alpha}{\beta(1-\alpha)} - \delta$$

Population Growth and Welfare Levels Given log utility, we can define an agent's lifetime utility as follows:

$$U_t = \log(\frac{w_t}{1+\beta}) + \beta\log(\frac{(1+r_{t+1})w_t\beta}{1+\beta})$$

Different rates of population growth will result in different steady state levels of k. The following relationship between agents' lifetime utility and the rate of population growth holds:

$$\frac{\partial U}{\partial n} = (1+\beta)\frac{\partial w}{\partial n}\frac{1}{w} + \frac{\partial r}{\partial n}\frac{\beta}{(1+r)}$$

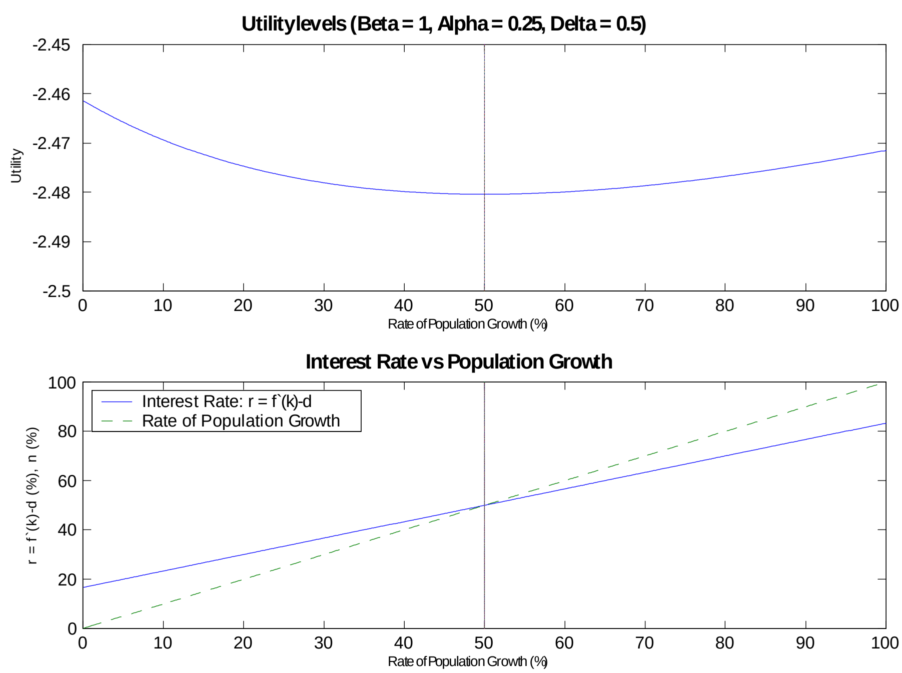
where both w and r are functions of n as defined above. The first term will always be negative, reflecting the impact of capital shallowing, while the second term will always be positive, as a result of the gains from increased investment returns. The following condition holds:

$$\frac{\partial U}{\partial n} = 0 \rightarrow n = \frac{(1+\beta)\alpha - \beta\delta(1-\alpha)}{\beta - 2\alpha\beta - \alpha}$$

This is the turning point in the social planner economy noted above. After the turning point, welfare increases with n. As an example, Figure 2 plots agents' lifetime utility and the equilibrium interest rate for a range of population growth rates when $\beta = 1$, $\alpha = 0.25$ and $\delta = 0.5$. Agents' steady state lifetime utililty is u-shaped in terms of n.

Within an OLG framework, it is usually not possible for agents to satisfy both of the social planner first order conditions due to incomplete markets. In fact, with log utility there is only one rate of population growth in which it is possible, in a two-period OLG framework, for agents to satisfy both of the these conditions in equilibrium: $n = \frac{(1+\beta)\alpha - \beta\delta(1-\alpha)}{\beta - 2\alpha\beta - \alpha}$.

Figure 2: The Relationship Between Lifetime Utility and Population Growth: Log Utility



Somewhat counterintuitively, agent's welfare is lower at the only rate of population growth that allows for the golden rule level of investment in equilibrium than at any other rate of population growth. As is shown in the bottom panel of figure 2, $r = n$ at the point of minimum utility. When n is less than this level, the distribution of income is such that it is not possible, without transfer payments, for agents to save enough such that $f\prime(k) = n + \delta$. When n is greater than this level, the economy is dynamically inefficient. Dynamic inefficiency occurs at low levels of k since n is growing faster than r.

There are two offsetting forces at play that result in this: capital deepening and investment income effects.

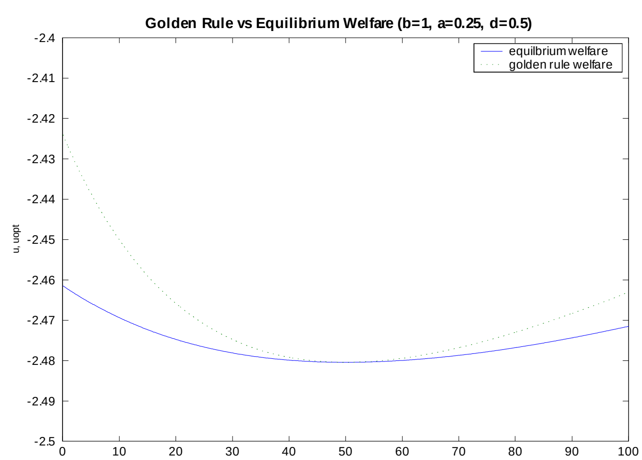
Increasing Consumption; Capital Shallowing As noted, there is a positive relationship between n and r. Therefore, as the rate of population growth increases, agents earn a lower wage but benefit from a higher return on their savings. As in the social planner economy, as the rate of population growth increases above n^{GR}, it is possible for agents' lifetime consumption profile to increase notwithstanding that the total resources available to society per worker are decreasing. With log utility, this occurs solely as a result of a rising interest rate since the saving rate remains constant. This reallocation is feasible, despite the reduction in per worker output since the elderly, whose consumption rises more rapidly than the consumption of the young falls, comprise an ever smaller proportion of the population as n increases. For agents, what is important is utility, not $c_1 + c_2$. However, the feasibility of increasing the overall consumption stream as a result of composition effects underlies the key result: that an increase in the population growth rate in a dynamically inefficient two-period OLG economy will exacerbate dynamic inefficiency while improving agents' welfare.

The Ratio of Consumption Across Periods In a competitive equilibrium OLG economy, today's young may save more than the golden rule level of capital since they know that they will not be able to trade with the young of tomorrow. If there are infinite generations, a social planner could arrange for transfer payments from today's young to today's old. In such a situation, it would make sense for today's young to make such a payment since they would receive a payment themselves the next period. In the presence of such payments, today's young would only need to save until $f\prime(k) = n + \delta$. In a world without transfer payments, it can be optimal for agents to save so much that the economy is dynamically inefficient. At levels of n greater than n^{GR}, the wage share of income is higher than is necessary to set $f\prime(k) = n + \delta$. The optimal situation would be to save so that there is golden rule level of capital and then implement transfer payments such that $\frac{c_2}{c_1} = (1+n)\beta$. In the absence of transfer payments, agents choose a compromise. They opt to save more than a social planner would choose, but by doing so, they are able to adjust the ratio $\frac{c_2}{c_1}$ closer to $(1+n)\beta$ than it would be if they set $f\prime(k) = n + \delta$.

Overall Effect: Comparison with Social Planner Outcome The sign of $\frac{\partial U}{\partial n}$ will depend upon the parameters and, specifically, whether the economy is dynamically inefficient or not. $\frac{\partial U}{\partial n} < 0$ when the economy is at a point where $n < n^{GR}$. Somewhat counterintuitively, $\frac{\partial U}{\partial n} > 0$ when the economy is dynamically inefficient. At levels of population growth higher than n^{GR}, agents skew their consumption to period 2 as a result of the higher interest rate thereby increasing their lifetime consumption stream even as society becomes poorer per worker. In a dynamically inefficient economy, the investment income effect of faster population growth outweighs the capital shallowing effect.

Samuelson and Deardorff showed that, in a social planner economy with log utility and Cobb-Douglas production, there is a welfare minimizing rate of population growth. This benchmark example is a useful point of comparison in order to understand why the turning point in the case of decentralized equilibrium outcomes occurs exactly at the point of dynamic in/efficiency cross over.

Figure 3: "Equilibrium" Lifetime Utility vs "Social Planner" Lifetime Utility: Log Utility



As Samuelson (1976) shows, the minimum of the golden rule welfare curve is also attainable as an equilibrium outcome. We know that the equilibrium outcome for each rate of population growth will never be better than the golden rule (social planner) outcome. Therefore, if the equilibrium welfare turning point is also the golden rule welfare turning point, we know that the two welfare curves are u-shaped and must have a point of tangency at their shared minimum as in Figure 3.

Samuelson and Deardorff focused on the golden rule outcome for each rate of population growth and, therefore, they didn't need to consider dynamic inefficiency. But since we can show that the golden rule and equilibrium welfare curves share the same minimum point, we can conclude that when an economy is dynamically efficient an increase in the rate of population growth will reduce agents' welfare but when an economy is dynamically inefficient, the reverse is true.

3.2 Case 2: Generalization of the Equilibrium Outcome

For notational simplicity, I assume zero depreciation for the generalization. Consumers maximize:

$$U = ((1-s)w) + \beta u(sw(1+r))$$

where s, w and r are all functions of n. When facing a given $w(n)$ and $r(n)$, agents will maximize utility by choosing s such that:

$$-u'((1-s)w(n))w(n) + \beta u'(sw(n)(1+r(n)))(1+r(n))w(n) = 0$$

Ths implies:

$$u'((1-s)w(n)) = \beta u'(sw(n)(1+r(n)))(1+r(n))$$

The returns to the factors of production, w and r, will be determined by the factor price frontier. Since r and w are both functions of n, it is possible to express the impact of a change of n on lifetime utility by calculating how lifetime utility changes with respect to either w or r. Using the envelope theorem, and having already maximized $U(s)$ with respect to s for a given r, I express the change in U with respect to $r(n)$ (and, implicitly, $w(n)$):

$$U'(r) = u'((1-s)w(r))(1-s)w'(r) + \beta u'(s(1+r)w(r))s[w(r) + (1+r)w'(r)]$$

From the optimal savings condition above, it follows that:

$$U'(r) = \beta u'(s(1+r)w(r))[(1+r)(1-s)w'(r) + sw(r) + s(1+r)w'(r)]$$

$$= \beta u'(s(1+r)w(r))[(1+r)w'(r) + sw(r)]$$

Since $u\prime(.) > 0$, $U'(r(n)) = 0$ if:

$$(1+r)w'(r) + sw(r) = 0$$

As noted earlier, the capital accumulation function implies that:

$$k = \frac{sw(r)}{1+n}$$

If we assume a CRS production function, then the following condition holds along the factor price frontier:

$$\frac{\partial w}{\partial r} = -k$$

Combining this with the capital accumulation function gives the following condition:

$$(1+n)w'(r) + sw(r) = 0$$

When combined with the earlier condition: $(1+r)w'(r) + sw(r) = 0$, this implies that $U\prime(r) = 0$ when $r = n$. Further, since $w\prime(r) < 0$:

$$U'(r) > 0 \rightarrow (1+r)w'(r) + sw(r) > 0 \rightarrow n > r$$

The same result holds if agents work in both periods.[1] Therefore, in the two period OLG model, neither the choice of utility function nor the pattern of life cycle earning affects the result. Importantly, if the elasticity of intertemporal substitution is low enough, the economy will never become dynamically inefficient in equilibrium. In such cases, agents' welfare declines monotonically with population growth, as in the infinitely lived agent model. Agents' lifetime welfare increases in n only if the economy becomes dynamically inefficient at some point.

4 Conclusion

In a two-period OLG model, there is only one competitive equilibrium outcome that is also a golden rule outcome. This is the welfare minimizing rate of population growth. If a dynamically inefficient two period OLG economy experiences an increase in population growth, the economy will become more dynamically inefficient, yet agents will be better off. This result is attributable to the u-shaped relationship between welfare and population growth when the elasticity of intertemporal substitution is high enough. Due to the fall in the dependency ratio, it is possible that as population growth increases, agents' lifetime consumption will increase even as output per worker decreases.

5 Bibliography

References

[1] Deardorff, Alan V., "The Optimum Growth Rate for Population: Comment", *International Economic Review*, Vol. 17, No.2, (1976), $531 - 538$.

[2] Samuelson, Paul A., "The Optimum Growth Rate for Population", *International Economic Review*, Vol. 16, No.3, (1975), $510 - 515$.

[1] The same result also holds for the three period model in which agents work in the first period.

[3] Samuelson, Paul A., "The Optimum Growth Rate for Population: Agreement and Evaluations", *International Economic Review*, Vol. 17, No.2, (1976), 516 – 525.