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REVISITING THE ORDERED FAMILY OF LORENZ CURVES

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ABSTRACT

Sarabia *et al.* (1999) present a basic model to create ordered families of Lorenz curves, along with basic theorems that describe the conditions for the models to satisfy the definition of the Lorenz curve. Using these basic models, they suggest a family which includes several well-known Lorenz models as special cases. This paper first shows that their basic theorems can be generalized. The paper then proceeds to propose additional families of Lorenz models. Finally the performance of some of the models is compared and it is shown that more efficient Lorenz models are possible with the assistance of our generalized result of the Sarabia *et al.* (1999) basic model.

JEL classification: D3; C5

Keywords: Lorenz curve; Gini index

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1. Introduction

The Lorenz curve of an income distribution is a functional relationship between the bottom p percent of individuals and their share of total income $L(p)$. Since income data are often grouped, rather than available at the individual level, it is not possible to obtain the relevant Lorenz curve of the distribution. Thus, special procedures need to be found to obtain an approximation of the Lorenz curve. There has been continuous interest in finding parametric Lorenz models to fit such grouped data in the literature in recent years (see Basmann *et al.* 1990; Cheong, 2002; Chotikapanich, 1993; Gupta 1984; Holm 1993; Kakwani 1980; Kakwani & Podder, 1973, 1976; Ogwang & Rao 1996, 2000; Ortega *et al.* 1991; Rao & Tam 1987; Rasche *et al.* 1980; Rossi 1985; Ryn & Slottje 1996; Sarabia 1997; Sarabia *et al.* 1999, 2001, 2005; Schader & Schmid 1994; Villasenor & Arnold, 1989). These models can also be used to estimate the Lorenz curve from micro-data of an income distribution, containing a series of individual observations on the underlying Lorenz curve.

Some models used to approximate the Lorenz curve in the literature do not in fact satisfy the definition of the Lorenz curve. Schader and Schmid (1994) list a few such models. Among such models, those developed by Kakwani and Podder (1976) and Kakwani (1980) may be the most representative. These models, nevertheless, continue to be used because their functional forms are simple, their parameter estimation can be simplified by linearization, their Gini index can be represented with simple formulae, and their performance has been shown to be good (see Ortega *et al.* 1991; Cheong, 2002). Recent research has concentrated on finding models which are satisfactory both in theory and in practice. These models need to be able to satisfy the definition of the Lorenz curve and have good performance for a variety of data sources (see, for example, Ogwang & Rao, 2000, Ryn & Slottje, 1996; Sarabia *et al.* 1999). These models cannot in general be estimated using linearization and may not necessarily have Gini formulae of a closed form.

Sarabia *et al.* (1999) suggest a basic model of the form $\tilde{L}(p) = p^{\alpha} L(p)^{\eta}$ and point out that a family of Lorenz curves can be obtained from it. They claim that the family can be ordered with respect to their parameters and the order corresponds to Lorenz dominance (see Atkinson 1970; Dasgupta *et al.* 1973; Davies & Hoy 1995; Shorrocks 1983). They describe the conditions for $\tilde{L}(p)$ to be a Lorenz curve and suggest a practical family that conforms exactly with the basic model. It is often difficult to estimate the parameters of Lorenz models in practice because most of them are highly nonlinear. They provide an algorithm to estimate the parameters with which one can avoid resorting to the nonlinear least square (NLS) method, which has been a common approach in the Lorenz model literature. It is well known that, in general, the more parameters that a model possesses, the more efficient the model may be (for a discussion of this point see, for example, Bates & Watts 1988). The fundamental premise of their basic model can be understood as embedding more parameters.

However, the usefulness of Sarabia *et al.*'s (1999) hierarchical order concept is limited and the order, defined in the set of Lorenz curves generated by varying the parameters of the models, is actually a partial one. Furthermore it can be shown that their condition for $\tilde{L}(p)$ to be a Lorenz curve is too restrictive. Nevertheless, in spite of these shortcomings, the basic model and the notion of a family of Lorenz curves are important contributions to the literature on which this paper seeks to build.

This paper is set out as follows. Section 2 reviews the results associated with the basic form and the ordered family of Lorenz curves given by Sarabia *et al.* (1999). Section 3 describes the generalization of their results and presents some applications of the generalized results. Section 4 introduces our family of Lorenz curves using the generalization of Sarabia *et al.*'s (1999) basic model. Section 5 presents some test estimations. Section 6 concludes the paper and makes suggestions for future research.

2. Review of the ordered family of Lorenz curves

We first introduce the definition of the Lorenz curve.

Definition. Assume $L(p)$ is defined and continuous in the interval $[0,1]$ with second derivative $L''(p)$. It is called a Lorenz curve if:

$$L(0)=0,\ L(1)=1,\ L'(0^+)\geq 0 \text{ and } L''(p)\geq 0 \text{ for all } p\in[0,1].$$

Sarabia *et al.* (1999) give two theorems to describe the condition for the basic form of $\tilde{L}(p)=p^{\alpha}L(p)^{\eta}$ to be a Lorenz curve which can be briefly written as a single result:

Theorem 1 *(Sarabia et al. 1999). If $L(p)$ is a Lorenz curve and $\eta\geq 1$, then*

$$\tilde{L}(p)=p^{\alpha}L(p)^{\eta}$$

is a Lorenz curve if one of the following conditions holds

(1) $\alpha=0$,

(2) $\alpha\geq 1$,

(3) $\alpha\in[0,1)$ *and* $L'''(p)\geq 0$ *for all* $p\in[0,1]$.

Thus, from a special parametric Lorenz model $L(p)$, three new models can be established. From the Lorenz curve associated with the classical Pareto distribution:

$$S_0(p)=1-(1-p)^{\beta} \qquad (1)$$

Sarabia *et al.* (1999) introduce the generalized Pareto family of Lorenz curves which encompasses the model in Equation (1) and the following three models

$$S_1(p) = p^{\alpha}[1-(1-p)^{\beta}]\,, \quad (2)$$

$$S_2(p) = [1-(1-p)^{\beta}]^{\eta}\,, \quad (3)$$

$$S_3(p) = p^{\alpha}[1-(1-p)^{\beta}]^{\eta}\,. \quad (4)$$

S_1 and S_2 are well-known Lorenz models suggested by Ortega *et al.* (1991) and Rasche *et al.* (1980) respectively. S_3 is a new model discussed further below.

The hierarchical order concept can be best explained by using the basic form. Following Sarabia *et al.* (1999), it is easy to see that, for example, if $\alpha_1 \le \alpha_2$, $\eta_1 \le \eta_2$ and $L(p)$ is a Lorenz curve, then $p^{\alpha_1}L(p)^{\eta_1} \ge p^{\alpha_2}L(p)^{\eta_2}$ holds for all $p \in [0,1]$. Therefore the distribution with α_1 and η_1 has higher welfare according to the Lorenz dominant criterion. However, it may be difficult to make such comparisons in general. For instance, our comparative result for the basic model here is premised upon an assumption that the term $L(p)$ on both sides of the inequality is the same or that $L(p)$ is simple enough so that further analysis is convenient. If, instead, $L(p)$ is itself a complicated multi-parametric function as in some of the models provided in the following discussion, the comparison may rarely be possible since the parameters of $L(p)$ will most probably vary when $\tilde{L}(p)$ is applied to different data.

Sarabia *et al.* (1999) claim that, because $S_0(p)$ is a Lorenz curve with $S_0'''(p) \ge 0$, $S_3(p)$ possesses the following property:

Theorem 2. *$S_3(p)$ is a Lorenz curve for any*

$$\alpha \ge 0,\ \beta \in (0,1] \text{ and } \eta \ge 1. \quad (5)$$

The conditions for the models in Equations (2) and (3) to be Lorenz curves given by Ortega *et al.* (1991) and Rasche *et al.* (1980) respectively are direct corollaries of this theorem. In the next section we show this result and theorem 1 can be generalized.

3. A useful theorem for Lorenz model building

The condition that $L'''(p) \geq 0$ in theorem 1 above is a severe restriction and can exclude many obvious cases. For example, take $L(p) = p^{\lambda}$ with $\lambda \in (1,2)$. $L(p)$ is clearly a Lorenz curve with $L'''(p) < 0$ for all $p \in (0,1]$. But $p^{\alpha} L(p)^{\eta} = p^{\alpha+\lambda\eta}$ is a Lorenz curve for any $\alpha \geq 0$ and $\eta \geq 1$. Take a more general example. Suppose $F(x)$ is an income distribution function with a continuously differential density $f(x)$ and a Lorenz curve $L(p)$. By the definition of the Lorenz curve, we have:

$$L(p) = \frac{1}{\mu} \int_0^x t f(t) \mathrm{d}t ,$$

where μ is the average income of the distribution and $p = F(x)$. Thus:

$$L'''(p) = -\frac{f'(x)}{\mu f(x)^3} .$$

Therefore $L'''(p) \geq 0$ implies that $f(x)$ is decreasing. This condition may only hold for some special cases, such as the Pareto density function. The condition $L'''(p) \geq 0$ is too demanding and may exclude many Lorenz curves. Fortunately, this condition is unnecessary. An alternative is possible, as the following theorem shows:

Theorem 3. *Assume $L(p)$ is a Lorenz curve. $\tilde{L}(p) = p^{\alpha} L(p)^{\eta}$ is a Lorenz curve for any $\alpha \geq 0$ and $\eta \geq 1$. Furthermore, if $L'''(p) \geq 0$ for all $p \in [0,1]$, then $\tilde{L}(p)$ is a Lorenz curve if $\alpha \geq 0$, $\eta \geq 1/2$ and $\alpha + \eta \geq 1$.*

Proof. Consider the first statement. Note

$$\tilde{L}'(p) = \alpha p^{\alpha-1}L(p)^{\eta} + \eta p^{\alpha}L(p)^{\eta-1}L'(p),$$

$$\frac{\tilde{L}''(p)}{p^{\alpha-2}L(p)^{\eta-2}} = \alpha(\alpha-1)L(p)^2 + \alpha\eta pL(p)L'(p)$$
$$+\eta p^2\left[(\eta-1)L'(p)^2 + L(p)L''(p)\right] + \alpha\eta pL(p)L'(p). \qquad (6)$$

$\tilde{L}(p)$ is a Lorenz curve when $\alpha \geq 1$. Assume $\alpha \in [0,1)$. It is sufficient for $\tilde{L}(p)$ to be a Lorenz curve if the sum of the first two terms on the right side of Equation (6) is non-negative, or

$$(\alpha-1)L(p) + \eta pL'(p) \geq 0. \qquad (7)$$

This inequality clearly holds since the function on the left side of the inequality is equal to zero at $p = 0$ and is increasing on $[0,1]$ from the derivative of the function.

Next consider the second statement. Note under the present assumptions that the inequality in Equation (7) still holds so that we only need to prove that the sum of the last two terms on the right side of Equation (6), is non-negative. Let the sum of the last two terms on the right hand side of Equation (6) be denoted as $g(p)$ after being divided by ηp for later reference. We only need to prove that:

$$g(p) = p\left[(\eta-1)L'(p)^2 + L(p)L''(p)\right] + \alpha L(p)L'(p) \geq 0 \text{ for all } p \in [0,1] \qquad (8)$$

holds for any $\eta \in [1/2, 1)$ and $\alpha \geq 0$ satisfying $\alpha + \eta \geq 1$. This inequality is true given the assumptions, because $g(p)$ is increasing on $[0,1]$ and satisfies $g(0) = 0$.

We now present four examples to demonstrate the usefulness of theorem 3 and, at the same time, point out some pitfalls in using this result.

Example 1. A generalization of $S_3(p)$.

As a first application of theorem 3, note that $S_0(p)$ is a Lorenz curve with $S_0'''(p) \geq 0$, thus we find immediately that theorem 2 can be generalized as:

Theorem 4. *$S_3(p)$ is a Lorenz curve if*

$$\alpha \geq 0,\ \beta \in (0,1],\ \eta \geq 1/2 \text{ and } \alpha + \eta \geq 1. \quad (9)$$

Note the condition $\alpha + \eta \geq 1$ in theorem 4 cannot be relaxed since if, to the contrary, $\alpha + \eta < 1$, then when we let $\beta = 1$, we get $S_3(p) = p^{\alpha+\eta}$ which is not a Lorenz curve.

Example 2. A mixed hybrid Lorenz model.

Ogwang and Rao (2000) suggest two hybrid methods to build Lorenz models. These are the weighted product and the convex combination of Lorenz models. In fact, $\tilde{L}(p)$ belongs to the former category. They find the convex combination model of

$$L_{OR}(p) = \delta p^{\alpha}\left[1-(1-p)^{\beta}\right] + (1-\delta)\frac{e^{\lambda p}-1}{e^{\lambda}-1},\ \delta \in [0,1]$$

is satisfactory, where one component is as depicted in Equation (2) and another is

$$L_C(p) = \frac{e^{\lambda p}-1}{e^{\lambda}-1},\ \lambda > 0$$

which is suggested by Chotikapanich (1993). Since both $S_0(p)$, which is associated with the Pareto distribution, and $L_C(p)$ satisfy $L'''(p) \geq 0$ on $[0,1]$, a different model or a Lorenz curve family can be established by using theorem 3:

$$L_{PC}(p) = p^{\alpha}\left\{\delta\left[1-(1-p)^{\beta}\right] + (1-\delta)\frac{e^{\lambda p}-1}{e^{\lambda}-1}\right\}^{\eta}$$

with parameter range

$$\alpha \geq 0,\ \beta \in (0,1],\ \lambda > 0,\ \delta \in [0,1],\ \eta \geq 1/2,\ \alpha + \eta \geq 1.$$

$L_{PC}(p)$ is a mixed hybrid Lorenz curve as both hybrid methods are used in its formation. Moreover, we can introduce an even more general mixed hybrid model:

$$G(p) = L_1(p)^{\alpha}\left(\delta L_2(p) + (1-\delta)L_3(p)\right)^{\eta}$$

where $L_1(p)$, $L_2(p)$ and $L_3(p)$ are all Lorenz curves. The weighted product and convex combination models proposed by Ogwang & Rao (2000) are special cases of $G(p)$.

Example 3. A generalization of the model proposed by Sarabia *et al.* (2001).

The above two examples show the usefulness of theorem 3 in creating parametric Lorenz models which possess the basic form. The present example shows that theorem 3 may still be too restrictive. While the condition $\alpha+\eta\geq 1$ is needed for $\tilde{L}(p)$ to be a Lorenz curve as demonstrated in the first example, it may be expected that the condition $\eta\geq 1/2$ can be further extended in some special cases. This is indeed the case. Note the second part of theorem 3 is true if $g(p)\geq 0$, as per the proof of theorem 3. Consider using $L_C(p)$ in place of $L(p)$. It can be shown that

$$g(p)=\frac{\lambda e^{\lambda p}}{(e^{\lambda}-1)^2}\left\{\, p\left[(\eta-1)\lambda e^{\lambda p}+(e^{\lambda p}-1)\lambda\right]+\alpha(e^{\lambda p}-1)\,\right\}\geq 0$$

holds for any $\eta\in[0,1)$ and $p\in[0,1]$, because the function between the braces is increasing and equal to zero at $p=0$. Thus we have obtained a Lorenz model:

$$L_S(p)=p^{\alpha}\left[\frac{e^{\lambda p}-1}{e^{\lambda}-1}\right]^{\eta},\ \alpha\geq 0,\ \lambda>0,\ \eta\geq 0 \text{ and } \alpha+\eta\geq 1.$$

This is the model that is recommended by Sarabia *et al*. (2001), but with the imposition that $\eta\geq 1$. This example shows the risk of simply relying on theorem 3 to determine the range of η and suggests that a good strategy is to check directly whether $g(p)\geq 0$ holds in some cases.

Example 4. A generalized model of both Kakwani and Podder (1973) and Gupta *et al.* (1980)

It may be more convenient to verify directly that $\tilde{L}(p)$ satisfies the definition of a Lorenz curve in some situations. Consider the Lorenz model $L_G(p) = pA^{p-1}$ with $A > 1$ suggested by Gupta *et al.* (1980). It is clear that $L_G(p)$ satisfies $L'''(p) \geq 0$. Thus theorem 3 implies that

$$L_1(p) = p^{\alpha+\eta} A^{\eta(p-1)}$$

is a Lorenz curve if $\alpha + \eta \geq 1$, $A > 1$ and $\eta \geq 1/2$. But, after rewriting $\alpha + \eta$ as α, it is straightforward to verify directly that

$$L_{KP}(p) = p^{\alpha} A^{\eta(p-1)}$$

satisfies the definition of the Lorenz curve if $\alpha \geq 1$, $A > 1$ and $\eta \geq 0$. This model encompasses both the model of Kakwani and Podder (1973), where A is fixed at $A = e$, and $L_G(p)$ as special cases.

4. A new family of Lorenz curves

The model suggested by Schader and Schmid (1994) is:

$$L_{SS}(p) = p^{\alpha}\left[1 - \eta p^{\gamma}(1-p)^{\beta}\right].$$

The fitted results of this model are impressive. However, it has the disadvantage that its admissible parameter range cannot be determined so that one cannot be certain if a fitted result satisfies the condition of the Lorenz curve. One option is to replace the term ηp^{γ} with $e^{-\gamma p}$ in order to create more satisfactory Lorenz models. Therefore, we first recommend a bi-parametric model:

$$H_0(p) = 1 - (1-p)^{\beta} e^{-\gamma p}. \qquad (10)$$

$H_0(p)$ is a generalization of the classical Pareto Lorenz curve. When it is used as a parametric Lorenz model, we find that it can compete with the models in Equations (2) or (3), and, in general, it performs much better than $S_0(p)$.

Following Sarabia *et al.* (1999) we suggest the following family of Lorenz curves which contains the model in Equation (10) and the following three models

$$H_1(p) = p^{\alpha}\left[1-(1-p)^{\beta}e^{-\gamma p}\right], \tag{11}$$

$$H_2(p) = \left[1-(1-p)^{\beta}e^{-\gamma p}\right]^{\eta}, \tag{12}$$

$$H_3(p) = p^{\alpha}\left[1-(1-p)^{\beta}e^{-\gamma p}\right]^{\eta}. \tag{13}$$

Next we address the essential condition for the models to be Lorenz curves. Note $H_i(0)=0$ and $H_i(1)=1$ for $i=0,\cdots,3$ if $\alpha \geq 0$, $\beta > 0$ and $\eta > 0$. For the functions specified in Equations (10) to (13) to be Lorenz curves, we need only find the condition for $H_3(p)$ as the other three specifications are special cases of $H_3(p)$.

Theorem 5. $H_3(p)$ *is a Lorenz curve if its parameters satisfy either*

$$\alpha \geq 0,\ \beta > 0,\ 0 \leq \beta+\gamma \leq \sqrt{\beta},\ \eta \geq 1, \tag{14}$$

or

$$\alpha \geq 0,\ \beta \in (0,1],\ 0 \leq \beta+\gamma \leq \sqrt{\beta},\ \eta \geq 1/2 \text{ and } \alpha+\eta \geq 1. \tag{15}$$

Proof. It is straightforward to verify that

$$H_0'(p) = \left[\beta+\gamma(1-p)\right](1-p)^{\beta-1}e^{-\gamma p},$$

$$H_0''(p) = \left\{\beta-\left[\beta+\gamma(1-p)\right]^2\right\}(1-p)^{\beta-2}e^{-\gamma p}.$$

Thus H_0 is a Lorenz curve for all (β,γ) satisfying $\beta > 0$ *and* $0 \leq \beta+\gamma \leq \sqrt{\beta}$, and the first part of theorem 3 implies $H_3(p)$ is a Lorenz curve under condition (14).

Note

$$\frac{H_0'''(p)}{(1-p)^{\beta-3}e^{-\gamma p}} = 2\gamma[\beta+\gamma(1-p)](1-p)$$

$$+\left\{\beta-[\beta+\gamma(1-p)]^2\right\}\left\{2-[\beta+\gamma(1-p)]\right\}.$$

Denote the function on the right side of this equation as $f(p)$. We can claim that $f(p)\geq 0$ and, consequently, $H_0'''(p)\geq 0$ for all $p\in[0,1]$ if (β,γ) satisfies:

$$\beta\in(0,1],\ 0\leq\beta+\gamma\leq\sqrt{\beta}\,. \tag{16}$$

The statement is true if $\gamma\geq 0$. Consider the case if $\gamma<0$. The condition in (16) implies $f'(p)\leq 0$ so that $f(p)$ is a decreasing function on $[0,1]$. But $f(1)=\beta(1-\beta)(2-\beta)\geq 0$ for any $\beta\in(0,1]$. Thus $f(p)\geq 0$ does hold for all $p\in[0,1]$ if condition (16) is true. Drawing on the second part of theorem 3, we have proven that $H_3(p)$ is a Lorenz curve if the condition specified in (15) is satisfied.

Since β is permitted to be any number satisfying $\beta>0$ and $0\leq\gamma+\beta\leq\sqrt{\beta}$ in condition (14), the extent for γ to vary is considerable given the admissible range of β. Note that the condition specified in (15) is obtained from that in (14) by a small expansion of η from $\eta\geq 1$ to $\eta\geq 1/2$. But this results in a substantial reduction in the admissible ranges of β and γ. However, our estimations suggest that the gain in efficiency more than offsets the loss of freedom attributed to β and γ. Because both $H_0(p)$ and $L_C(p)$ satisfy $L'''(p)\geq 0$, another seemly complicated mixed hybrid family, which is analogous to $L_{PC}(p)$ above, can be given as

$$L_{HC}(p)=p^{\alpha}\left\{\delta\left[1-(1-p)^{\beta}e^{-\gamma p}\right]+(1-\delta)\frac{e^{\lambda p}-1}{e^{\lambda}-1}\right\}^{\eta}$$

This result can be expressed in the form of the following theorem:

Theorem 6. $L_{HC}(p)$ *is a Lorenz curve if*

$$\alpha \ge 0,\ \beta \in (0,1],\ 0 \le \beta + \gamma \le \sqrt{\beta},\ \lambda > 0,\ \delta \in [0,1],\ \eta \ge 1/2 \text{ and } \alpha + \eta \ge 1. \tag{17}$$

The parameter ranges specified in (14), (15) or (17), as well as that associated with $L_{PC}(p)$, contain complicated conditions. Their enforcement is convenient in practice if a NLS algorithm is used to estimate the parameters. Unfortunately, there is not a Gini formula of closed form for the models in (12) or (13) when $\gamma < 0$, and none at all for $L_{PC}(p)$ and $L_{HC}(p)$. The Gini indices for the models in (10) and (11) are:

$$G_0 = -1 + 2\frac{\Gamma(\beta+1)}{\Gamma(\beta+2)}\ {}_1F_1(1, \beta+2, -\gamma),$$

$$G_1 = \frac{\alpha-1}{\alpha+1} + 2\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\ {}_1F_1(\alpha+1, \alpha+\beta+2, -\gamma)$$

respectively. If $\gamma \ge 0$, the Gini indices for the models in Equations (12) and (13) are:

$$G_2 = 1 - 2\sum_{i=0}^{\infty}(-1)^i C_i(0, \beta, \eta),$$

$$G_3 = 1 - 2\Gamma(\alpha+1)\sum_{i=0}^{\infty}(-1)^i C_i(\alpha, \beta, \eta)$$

respectively. In these expressions, $\Gamma(\cdot)$ is the gamma function, ${}_1F_1(\cdot)$ is the confluent hypergeometric function (on special functions, see Wang, 1989) and

$$C_i(\alpha, \beta, \eta) = \frac{\eta(\eta-1)\cdots(\eta-i+1)}{i!}\frac{\Gamma(i\beta+1)}{\Gamma(i\beta+\alpha+2)}\ {}_1F_1(\alpha+1, i\beta+\alpha+2, -i\gamma).$$

The Gini index for the model in Equation (11) is G_1 because:

$$\int_0^1 p^{\alpha}(1-p)^{\beta}e^{-\gamma p}\mathrm{d}p = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\ {}_1F_1(\alpha+1, \alpha+\beta+2, -\gamma)$$

and

$$G_1 = 1 - 2\int_0^1 p^{\alpha}\left(1-(1-p)^{\beta}e^{-\gamma p}\right)\mathrm{d}p \ .$$

Letting $\alpha = 0$ and noting $\Gamma(1) = 1$, one obtains a Gini index G_0 for Equation (10).

Note:

$$(1-x)^{\eta} = \sum_{i=0}^{\infty}(-1)^{i}\,\frac{\eta(\eta-1)\cdots(\eta-i+1)}{i!}\,x^{i}$$

if η is neither a positive integer nor zero so that

$$p^{\alpha}\left[1-(1-p)^{\beta}e^{-\gamma p}\right]^{\eta} = \sum_{i=0}^{\infty}(-1)^{i}\,\frac{\eta(\eta-1)\cdots(\eta-i+1)}{i!}\,p^{\alpha}(1-p)^{i\beta}e^{-i\gamma p} \ .$$

The right side series converges uniformly on $[0,1]$ if $\gamma \geq 0$. Therefore the integral of the function on the left side is equal to the sum of the integrals of the terms on the right-hand side, so that G_3 is the Gini index of the model in Equation (13). If $\alpha = 0$ we get the Gini formula G_2 for the model in Equation (12).

5. Illustrative estimations

In this section some estimations are presented. Following Sarabia *et al.* (1999),

$$\text{MSE}=\frac{1}{n}\sum_{i=1}^{n}\left[L_i - L(p_i;\hat{\tau})\right]^2 ,$$

$$\text{MAE}=\frac{1}{n}\sum_{i=1}^{n}\left|L_i - L(p_i;\hat{\tau})\right| ,$$

$$\text{MAS}=\max_{1\leq i\leq n}\left|L_i - L(p_i;\hat{\tau})\right|$$

are used as error measures, where $L(p,\tau)$ is a Lorenz model with parameter vector τ, $\hat{\tau}$ is the least square estimator of τ and n is the sample size.

5.1. Computational considerations

We use the Levenberg-Marquardt algorithm (Dennis & Schnabel, 1983) to solve the NLS problem of minimizing the sum of residual squares (SRS) to obtain parameter estimates for our models. It is well-known that variable transformations can enforce constraints on the values of the parameters, which eliminates the need to invoke constraint optimization methods, and can improve convergence in nonlinear regressions (see, for example, Bates and Watts, 1988). To facilitate the parameter estimation of $S_3(p)$ with condition (9), we introduce a parameter transformation:

$$\begin{cases} \alpha = (1/2+\zeta^2)\sin^2\xi \\ \beta = \sin^2\theta \\ \eta = 1/2 + (1/2+\zeta^2)\cos^2\xi \end{cases} \tag{18}$$

and estimate (ζ,ξ,θ). The parameter estimation $(\hat{\alpha},\hat{\beta},\hat{\eta})$ obtained from (18) will make the corresponding response function satisfy the condition of the Lorenz curve given in (9) so that the constraints on the parameters are effectively imposed. We find that a satisfactory solution to the NLS problem can often be obtained if

$$\zeta = 1/2,\ \xi = \arcsin\sqrt{1/2},\ \theta = \arcsin\sqrt{1/2} \tag{19}$$

are used as starting values. Adding a function relation for parameter γ to (18), we suggest the following transformation in the estimation of $H_3(p)$ with condition (15):

$$\begin{cases} \alpha = (1/2+\zeta^2)\sin^2\xi \\ \beta = \sin^2\theta \\ \gamma = \sqrt{\beta}\sin^2\omega - \beta \\ \eta = 1/2 + (1/2+\zeta^2)\cos^2\xi \end{cases}$$

A satisfactory solution can often be produced if the starting values are taken as:

$$\zeta = 1/2,\ \xi = \arcsin\sqrt{1/2},\ \theta = \arcsin\sqrt{1/2},\ \omega = \arcsin\sqrt{1/2}\ . \tag{20}$$

Note the parameter ranges of $L_{PC}(p)$ and $L_{HC}(p)$ given in the last two sections can be enforced in an analogous manner. The fitted results below for the data used in our estimation depend upon the programming codes we are using. We have used a special routine which can systematically scan the parameter space to find appropriate starting values. While all the fitted results for $S_3(p)$ with condition (9) and $H_3(p)$ given below are obtained with the starting values given in (19) and (20) respectively, we find that the special search routine becomes essential when estimating $L_{PC}(p)$ and $L_{HC}(p)$.

5.2. Estimation Examples

We perform two tests. Denote $S_3(p)$ with condition (5) as $S_{3,(5)}$; $S_3(p)$ with condition (9) as $S_{3,(9)}$; and $H_3(p)$ with condition (15) as $H_{3,(15)}$.

5.2.1. Tests of $S_{3,(5)}$, $S_{3,(9)}$ and $H_{3,(15)}$.

Begin with the first test designed to compare $S_{3,(5)}$, $S_{3,(9)}$ and $H_{3,(15)}$. The data used in the test are income distributions for a range of countries from Shorrocks (1983). The results are displayed in Tables 1-3, where MSE values have been multiplied by 10^6 so that the largest MSE among all the cases is about 162×10^{-6} for the Indonesian data when $S_{3,(5)}$ is used. The Gini indices are given in the last column of the three tables. The Gini indices of the models are calculated using different methods. Sarabia *et al.* (1999) give a formula for $S_{3,(5)}$ and $S_{3,(9)}$. The Gini indices for H_0 and H_1 can be calculated using G_0 and G_1. However, only when $\gamma > 0$ can G_2 and G_3 be used to calculate the Gini indices of H_2 and H_3. Numerical integral is used for all other situations. The differences in the indices for each country are very small between the two models. All the fitted curves satisfy the definition of the Lorenz curve, since the transformations in the last sub-section are used in the estimation.

Table 1 shows the results of $S_{3,(5)}$. The model switches between $S_1(p)$ and $S_2(p)$ for most cases, although the actual values of η are not exactly unity and α is not exactly zero for the displayed values due to limitations on computational power. For the cases where the parameter estimation of η is equal to unity, it is reasonable to conclude that it may be the lower limit of $\eta \geq 1$ that prevents the SRS from declining further. The results in Table 2, which contain the results of $S_{3,(9)}$ but do not contain those countries, for which the results for $S_{3,(9)}$ are the same as that of $S_{3,(5)}$ to avoid repetition, shows that this conclusion is correct. It can be seen that the error deductions are quite substantial in the cases of India and Indonesia.

The results for $H_{3,(15)}$ in Table 3 are particularly good. The three errors for almost every case are satisfactorily small. The model shifting phenomenon can also be observed and there are as many as seven out of the nineteen cases where the estimation of α is equal to zero. This implies that $H_2(p)$ as defined in Equation (12) may be a suitable Lorenz model in practice. Compared with the results given by Sarabia. *et al.* (1999), our fitted results for $S_{3,(5)}$ in Table 1 for the Brazil and Swedish data are slightly better, reflecting the use of different algorithms.

5.2.2. Tests of six models presented in this paper

In this section we test the six models $S_{3,(5)}$, $S_{3,(9)}$, $H_{3,(15)}$, L_{OR}, L_{PC} and L_{HC}, which we suggest provide better performance in practice. Data on income distribution for the United States in 1977 and 1990 from Ryn & Slottje (1996) are used in this test. These data are also employed by Sarabia *et al*. (1999). The income distribution data for the United States from 1977 to 1983 can be found in Basmann *et al.* (1993). Ogwang & Rao (2000) use the data of 1977 to test their hybrid models.

The results are presented in Tables 4-7. Table 5 and Table 7 display the parameter estimates of the models, from which it can be seen that every curve obtained satisfies the definition of the Lorenz curve. Table 4 and Table 6 give absolute errors at each sample point for the models along with the three error measures for the two datasets respectively. The population and income shares are presented in first the two columns. The empirical Gini for the United States in 1977 is 0.3682 and for the United States in 1990 is 0.4325, which are listed on the last row of Tables 4 and 6 and are from Ryn & Slottje (1996). Other Ginis having five decimal places in the last line of the tables are calculated using the above alternative formulae or numerical integral. Consistent with Tables 1-3, the MSE values in the tables are the results of the actual values multiplied by 10^6 to save space. The estimated results for each year for $S_{3,(5)}$ and $S_{3,(9)}$ are the same. The slight differences in parameters reflect that we have used different starting values for the two models. Numbers in bold are used to indicate the smallest absolute error across the models after excluding L_{HC}. For the data given, it can be concluded that, relative to the other models, L_{HC} performs the best. L_{PC} performs second-best, but the performance of $H_{3,(15)}$ is only marginally inferior to L_{PC}.

6. Conclusion

This paper shows that the basic Lorenz model of $\tilde{L}(p) = p^{\alpha} L(p)^{\eta}$ and the concept of the ordered family of Lorenz curves, developed by Sarabia *et al.* (1999), are important ideas, although the condition imposed by Sarabia *et al.* (1999) for the models to be Lorenz curves proves to be restrictive. Theorem 3 in this paper ensures that we can construct new

Lorenz models from almost any parametric function $L(p)$ so long as it satisfies the condition of the Lorenz curve. Moreover, we can find even better performing Lorenz models if the condition $L'''(p) \geq 0$ is satisfied.

Based on theorem 3, model $S_3(p)$ in Sarabia *et al.* (1999) is generalized by extending its parameter range. The model with the new range proves to be better than both the original and its sub-models. Several other models are proposed using the theorem, among which $H_3(p)$ should be seen as the most important. Its form is relatively simple and its performance is satisfactory. Moreover, it forms a vital component in constructing the mixed hybrid model of L_{HC}. Our estimations suggest that L_{HC} may be an extremely efficient Lorenz model in practice.

Further research could use theorem 3 developed in this paper to find other $L(p)$, in order to create more efficient Lorenz models with the basic form $\tilde{L}(p)$. $L_S(p)$ introduced in Section 3 is an example that the basic form with $\eta \geq 1/2$ can be extended to $\eta \geq 0$ while remaining a Lorenz curve. Future research could find other models which are both efficient and have such larger admissible ranges of η.

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Table 1: Fitted results of $S_{3,(5)}$ for the Shorrocks (1983) data

	α	β	η	MSE$\times 10^6$	MAE	MAS	Gini
Brazil	0.5447	0.2833	1.0000	7.1796	0.0024	0.0046	0.6371
Columbia	0.6429	0.3895	1.0000	2.2899	0.0011	0.0033	0.5572
Denmark	0.0000	0.7812	1.6160	16.5340	0.0031	0.0099	0.3673
Finland	0.4893	0.6442	1.3450	3.4955	0.0016	0.0034	0.4707
India	0.2305	0.4246	1.0000	65.2760	0.0068	0.0163	0.4571
Indonesia	0.0219	0.3888	1.0000	164.1527	0.0110	0.0242	0.4452
Japan	0.0000	0.7235	1.3330	3.8403	0.0017	0.0029	0.3114
Kenya	0.3046	0.2678	1.0000	12.3318	0.0031	0.0053	0.6233
Malaysia	0.5720	0.4793	1.0869	0.9559	0.0008	0.0022	0.5113
Netherlands	0.5951	0.5558	1.0386	1.3382	0.0008	0.0029	0.4479
N Zealand	0.5878	0.6991	1.0488	3.8186	0.0014	0.0051	0.3692
Norway	0.6334	0.7068	1.0000	13.5501	0.0028	0.0087	0.3594
Panama	0.6194	0.5429	1.0000	1.5152	0.0009	0.0034	0.4466
Sri Lanka	0.0000	0.6965	1.5651	6.7435	0.0023	0.0043	0.4101
Sweden	0.0000	0.7324	1.5712	3.0472	0.0015	0.0035	0.3867
Tanzania	0.2474	0.3450	1.0096	8.8287	0.0027	0.0044	0.5389
Tunisia	0.0000	0.6230	1.7209	89.0902	0.0083	0.0161	0.5083
UK	0.4037	0.6051	1.0000	6.5033	0.0021	0.0046	0.3619
Uruguay	0.0000	0.7053	1.9281	40.2026	0.0054	0.0120	0.5000

Table 2: Cases where fitting is improved when $S_{3,(9)}$ is used for the Shorrocks (1983) data

	α	β	η	MSE $\times 10^6$	MAE	MAS	Gini
Brazil	0.6772	0.2622	0.9109	6.5162	0.0023	0.0042	0.6376
Columbia	0.8471	0.3504	0.8488	0.9197	0.0008	0.0022	0.5579
India	0.8875	0.2573	0.5000	6.0016	0.0020	0.0054	0.4590
Indonesia	0.6871	0.2235	0.5000	33.9401	0.0050	0.0115	0.4471
Kenya	0.4128	0.2506	0.9264	11.7341	0.0031	0.0050	0.6237
Norway	1.1811	0.5579	0.5000	9.2029	0.0021	0.0077	0.3601
Panama	0.6527	0.5362	0.9715	1.4968	0.0009	0.0033	0.4467
UK	0.8902	0.4752	0.5768	0.9024	0.0007	0.0026	0.3630

Note: The fitted results of those countries employing the Shorrocks (1983) data reported in Table 1, but not in this table are the same as $S_{3,(5)}$.

Table 3: Fitted results of $H_{3,(15)}$ for the Shorrocks (1983) data

	α	β	γ	η	MSE $\times 10^6$	MAE	MAS	Gini
Brazil	0.7116	0.2147	-0.1775	0.5689	3.0545	0.0016	0.0033	0.6374
Columbia	1.0165	0.3064	-0.1437	0.5846	0.5900	0.0006	0.0015	0.5579
Denmark	0.0000	0.8150	-0.2097	1.4158	8.5029	0.0024	0.0060	0.3655
Finland	0.0000	0.6875	0.0976	1.9123	2.5026	0.0014	0.0028	0.4709
India	0.9759	0.2262	0.0961	0.5000	1.2596	0.0010	0.0023	0.4600
Indonesia	0.8650	0.1597	0.1923	0.5000	2.6102	0.0014	0.0030	0.4487
Japan	0.0339	0.7418	-0.1303	1.2008	0.0967	0.0003	0.0005	0.3104
Kenya	0.4525	0.2012	-0.1686	0.5745	5.6715	0.0022	0.0039	0.6235
Malaysia	0.6194	0.4722	-0.0201	1.0295	0.9511	0.0008	0.0022	0.5113
Netherl.	0.1055	0.6171	0.1140	1.5903	1.1180	0.0009	0.0024	0.4479
N Zealand	0.0000	0.7596	0.0857	1.6891	3.0568	0.0013	0.0045	0.3694
Norway	0.9852	0.5996	0.1747	0.7576	7.2789	0.0019	0.0069	0.3605
Panama	0.0539	0.6131	0.1418	1.6489	1.1348	0.0008	0.0025	0.4467
Sri Lanka	0.0000	0.7210	-0.1763	1.3859	0.7277	0.0007	0.0017	0.4087
Sweden	0.0000	0.7449	-0.0784	1.4951	1.9078	0.0011	0.0030	0.3862
Tanzania	0.4304	0.2792	-0.2219	0.5696	2.7615	0.0015	0.0027	0.5386
Tunisia	0.0000	0.6470	-0.4335	1.1464	27.4604	0.0046	0.0099	0.5050
UK	0.5936	0.5422	0.1403	0.9162	0.6798	0.0007	0.0019	0.3630
Uruguay	0.0000	0.7417	-0.4737	1.2917	8.8744	0.0025	0.0059	0.4969

Table 4: Absolute errors of fitted Lorenz curves based on US 1977 income distribution data

p	Emp. $L(p)$	$S_{3,(5)}$	$S_{3,(9)}$	$H_{3,(15)}$	L_{OR}	L_{PC}	L_{HC}
0.10	0.0180	0.00186	0.00185	**0.00073**	0.00188	0.00172	0.00025
0.20	0.0528	0.00193	0.00190	**0.00033**	0.00169	0.00155	0.00010
0.30	0.1015	0.00092	0.00089	0.00046	**0.00032**	**0.00032**	0.00020
0.40	0.1644	**0.00010**	0.00014	0.00083	0.00095	0.00080	0.00026
0.50	0.2424	0.00048	0.00052	**0.00038**	0.00126	0.00107	0.00008
0.60	0.3364	**0.00037**	0.00042	0.00048	0.00061	0.00054	0.00033
0.70	0.4481	**0.00021**	0.00025	0.00105	0.00056	0.00036	0.00007
0.80	0.5814	**0.00025**	0.00028	0.00083	0.00145	0.00108	0.00039
0.90	0.7459	0.00054	0.00056	**0.00038**	0.00043	0.00039	0.00003
0.91	0.7649	0.00059	0.00061	0.00056	**0.00012**	0.00014	0.00005
0.92	0.7846	0.00060	0.00061	0.00070	0.00019	**0.00011**	0.00007
0.93	0.8052	0.00045	0.00046	0.00069	0.00037	**0.00024**	0.00017
0.94	0.8266	**0.00035**	0.00036	0.00072	0.00063	0.00047	0.00010
0.95	0.8491	**0.00017**	**0.00017**	0.00067	0.00082	0.00065	0.00004
0.96	0.8731	**0.00028**	**0.00028**	0.00033	0.00073	0.00059	0.00013
0.97	0.8989	0.00094	0.00094	**0.00024**	0.00038	0.00031	0.00026
0.98	0.9276	0.00227	0.00227	0.00153	0.00078	**0.00070**	0.00007
0.99	0.9596	0.00324	0.00324	0.00257	0.00193	**0.00160**	0.00014
MSE $\times 10^6$		1.46991	1.46979	0.84608	1.01381	0.73348	0.03434
MAE		0.00086	0.00087	0.00075	0.00084	0.00070	0.00015
MAS		0.00324	0.00324	0.00257	0.00193	0.00172	0.00039
Gini	0.3682	0.36899	0.36893	0.36862	0.36899	0.36888	0.36822

Note: model L_{HC} is excluded when absolute errors are compared. The empirical Gini is from Ryn & Slottje (1996).

Table 5: Parameter estimation of Lorenz models based on US 1977 income distribution data.

Parameter	$S_{3,(5)}$	$S_{3,(9)}$	$H_{3,(15)}$	L_{OR}	L_{PC}	L_{HC}
α	0.000000	0.000001	0.000000	0.643391	0.000000	0.609640
β	0.784050	0.784014	0.792222	0.734286	0.801389	0.816271
γ	---	---	-0.065438	---	---	-0.302703
λ	---	---	---	8.780359	10.049630	27.038897
η	1.628063	1.627774	1.559094	---	1.628601	0.890336
δ	---	---	---	0.951858	0.985251	0.964103

Table 6: Absolute errors of fitted Lorenz curves based on US 1990 income distribution data

p	Emp. $L(p)$	$S_{3,(5)}$	$S_{3,(9)}$	$H_{3,(15)}$	L_{OR}	L_{PC}	L_{HC}
0.10	0.0122	0.00233	0.00233	**0.00063**	0.00198	0.00170	0.00007
0.20	0.0379	0.00273	0.00274	**0.00003**	0.00175	0.00138	0.00008
0.30	0.0770	0.00212	0.00213	0.00050	0.00058	**0.00035**	0.00001
0.40	0.1303	0.00094	0.00095	**0.00072**	0.00079	0.00076	0.00001
0.50	0.1996	**0.00007**	**0.00007**	0.00031	0.00138	0.00111	0.00006
0.60	0.2870	0.00072	0.00072	**0.00050**	0.00083	0.00054	0.00005
0.70	0.3956	0.00107	0.00107	0.00113	**0.00059**	0.00060	0.00010
0.80	0.5311	0.00117	0.00118	**0.00095**	0.00182	0.00144	0.00001
0.90	0.7062	0.00128	0.00130	0.00077	0.00019	**0.00000**	0.00002
0.91	0.7272	0.00111	0.00112	0.00084	**0.00007**	0.00017	0.00004
0.92	0.7491	0.00088	0.00090	0.00088	**0.00032**	0.00034	0.00003
0.93	0.7721	0.00052	0.00054	0.00079	0.00048	**0.00042**	0.00006
0.94	0.7962	**0.00018**	0.00020	0.00072	0.00070	0.00056	0.00006
0.95	0.8218	0.00033	**0.00032**	0.00047	0.00074	0.00054	0.00014
0.96	0.8493	0.00112	0.00110	**0.00006**	0.00048	0.00027	0.00009
0.97	0.8792	0.00219	0.00217	0.00092	**0.00017**	0.00034	0.00015
0.98	0.9117	0.00289	0.00288	0.00150	0.00071	**0.00069**	0.00004
0.99	0.9487	0.00313	0.00312	0.00183	0.00135	**0.00095**	0.00004
MSE $\times 10^6$		2.75533	2.75495	0.75364	1.01064	0.66419	0.00501
MAE		0.00138	0.00138	0.00075	0.00083	0.00068	0.00006
MAS		0.00313	0.00312	0.00183	0.00198	0.00170	0.00015
Gini	0.4325	0.43357	0.43357	0.43284	0.43337	0.43320	0.43251

Note: model L_{HC} is excluded when absolute errors are compared. The empirical Gini is from Ryn & Slottje (1996).

Table 7: Parameter estimation of Lorenz models based on US 1990 income distribution data

Parameter	$S_{3,(5)}$	$S_{3,(9)}$	$H_{3,(15)}$	L_{OR}	L_{PC}	L_{HC}
α	0.000001	0.000001	0.000000	0.796917	0.000000	0.923089
β	0.756785	0.756837	0.770043	0.684324	0.768767	0.710755
γ	---	---	-0.127006	---	---	-0.570557
λ	---	---	---	8.070927	6.477915	20.914805
η	1.798009	1.798125	1.642049	---	1.765272	0.557075
δ	---	---	---	0.917923	0.972178	0.932815