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**FINITE LIFE EXPECTANCY AND THE AGE-DEPENDENT VALUE OF A
STATISTICAL LIFE**

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Abstract

In this short paper, we investigate the behavior of the age-dependent value of a statistical life (VSL) within a lifecycle framework with a *finite* maximal possible lifespan. Some existing results, obtained under the unrealistic assumption of an infinite life expectancy, are reversed. In particular, we show that when the market interest rate is equal to (or less than) the sum of age-specific mortality rate and the discounting rate in time preference at any age over the remaining lifetime, then VSL declines. We also show that an inverted-U shape of VSL profile over the life cycle emerges under realistically plausible circumstances. An innovation is that we characterize the changes in optimal consumption and instantaneous utility with age, showing that such changes are proportionate to the difference between the sum of age-specific mortality rate and the discounting rate in time preference and the market interest rate, which may prove to be useful in addressing other issues related to VSL.

JEL Codes: J17, D91.

Keywords: Value of life; life expectancy; interest rates; time preference; mortality.

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Finite life expectancy and the age-dependent value of a statistical life

In this short paper, we investigate the behavior of the age-dependent value of a statistical life (VSL) within a lifecycle framework with a *finite* maximal possible lifespan. Under the circumstance the maximal possible lifespan is finite (say 140 years old, to be naively optimistic), the life expectancy is always limited too. Some existing results, obtained under the unrealistic assumption of an infinite life expectancy, are reversed.

Age-specific VSL, and its lifecycle patterns, have profound implications for both business practice and policy making.¹ Some possible lifecycle patterns of VSL have been studied. On the one hand, many people may intuitively think that life is more valuable for the young as they expect to live longer. Arthur (1981), among others, has long before theoretically shown that VSL, measured in dollars, declines with age under certain circumstances. Dillingham, Miller and Levy (1996) take a WTP approach in examining data collected from the Australia labor market and find that the value of a remaining worklife declines with age. Of practical significance is that European Union (2001) goes as far as to propose to take this result as a policy guideline. On the other hand, lifecycle patterns of VSL can be far more complicated than the monotone declining curves with age. Indeed, an inverted-U shape of the age-profile of VSL have been noticed in Jones-Lee (1976), Shepard and Zeckhauser (1984), Rosen (1988), Kniesner, Viscusi and Ziliak (2004) and Ehrlich and Yin (2005).

The most sophisticated and insightful analysis on the age dependency of VSL, however, is due to Johansson (2002). One of Johansson's (2002) two major contributions is that the widely held belief that VSL generally declines with age is theoretically premature and that the age-profile of VSL crucially depends on the market interest rate,

¹ See Viscusi and Aldy (2003) for a critical literature review of VSL and especially Section 8 therein for the age-related values of life.

the time preference and age-specific mortality rates.² Of particular interest to our purpose here is that Johansson, using a lifecycle model with an infinite maximal lifespan, meticulously investigates the crucial role played by the foresaid factors in shaping the age-profile of VSL. It is shown that VSL may increase, decrease, or even display an inverted u-shape as one ages, and that if the market interest rate equals the sum of the time preference and mortality rates throughout the life cycle then a constant VSL follows.

Nobody, of course, can live infinitely long, and even a 99-year lifespan appears fairly luxurious to most humans. It is true that in dynamic models an infinite decision horizon is often assumed to facilitate technical treatment. For analysis of the age-dependency of VSL, however, it is precisely a (statistical) life – both in quantity and in quality, the so called quality adjusted life years – that is to be valued. The maximal possible lifespan should, and does, matter (as shown below). Making use of a lifecycle framework that accommodates a finite maximal possible lifespan, this paper explores the implications of a limited lifespan for the age-dependency of VSL. Several existing results turn out to be no longer valid under such a more realistic set-up.

We adopt the willingness-to-pay (WTP) approach, the dominating one in the rapidly growing literature on valuing life and health. Two concepts of the value of a life must be distinguished from each other. The *utility value* of life is simply the utility expected to be enjoyed in the rest of one's life, often referred to as quality-adjusted life years (QALY) in the health economics literature. The *dollar value* of life (VSL) reflects one's WTP for a reduction in the risk of death. For a rational expected utility maximizer, the utility and the dollar values of life are related in a definite manner. If we denote the utility and dollar values of (remaining) life at age a as $V(a)$ and $D(a)$ respectively and the current-value

² The other point, going beyond the scope of the present study, is that to empirically make an accurate estimate of VSL one has to, in principle, resort to a true blip in reduction of the mortality rate, i.e. a Dirac-delta drop in mortality rates, as remarked by Rosen (1988, p.29) in his theoretical analysis.

of life-time wealth at age a as $Y(a)$, we have $D(a) = V(a)/[\partial V(a)/\partial Y(a)]$ (see, e.g. Linnerooth 1979 and Ng 1992).

We organize the paper as follows. The next section introduces the model and conducts some preliminary analysis. Section 2 presents our main findings and Section 3 contains further discussions, especially on how our findings are related to the existing studies.

1. The model

Like in Johansson (2002), we consider a life cycle model in which the individual at any age maximizes her/his expected sum of discounted instantaneous utilities, deriving from consumption (c), over the rest of the lifespan. The utility function $u(c)$, assumed to be twice differentiable, increasing and strictly concave, maps all the possible non-negative amounts of consumptions into $R^+ \equiv [0, \infty)$.

An individual has given preferences and a maximal possible lifespan. She can freely lend and borrow at the same market (real) interest rate $r > 0$. The age-specific cumulative probability of survival is denoted as $s(t)$, $s(t) \in [0,1]$, $\forall t \in (0,T)$, which is determined by the age-specific mortality rates $\delta(t)$ as such $s(t) = s(0)e^{-\int_0^t \delta(s)ds}$ ($\delta(t) \geq 0, \forall t \in (0,T)$), and satisfies $s(0) = 1$ and $s(T) = 0$, implying that T is the maximal possible lifespan.³ The conditional probability of surviving until age t of a person aged a ($t \geq a$) is $s_a(t) = s(t)/s(a)$ (Rosen 1988, p. 293). Thus, the problem for an individual at age a is,

$$V(a) = \text{Max} \int_a^T u(c_t) s_a(t) e^{-[\theta(t)-\theta(a)]} dt \quad \text{subject to} \quad Y(a) = \int_a^T c_t e^{-r(t-a)} dt \quad (1)$$

³ The age-dependent survival rates are treated in our model as exogenous parameters, hence possible investment on health to reduce mortality is assumed away. Life insurance is also absent, partly because the major predictions of the perfect insurance life cycle models of VSL are not empirically supported (see, e.g. Kniesner et al 2004).

where $\theta(t)$ is the time preference parameter, assumed to be differentiable throughout for technical convenience. Note $\theta(t)$ may be interpreted as the cumulated depreciation in the capacity to derive utility at age t . That is, $\theta'(t)$ refers to the depreciation rate (with age) at time t of the capacity to derive utility and is allowed to be different at different ages.⁴ Either way one would like to take in interpreting $\theta'(t)$ would not affect our analysis below. Since possible controversies centred round the interpretation, as well as the appropriate treatment of time discounting, are not the major concern of this paper, we content ourselves with rather loosely referring to the term $\theta'(t)$ as the time preference parameter and the depreciation rate interchangeably. We set $\theta(0) = 0$, taking the state at the beginning of the life cycle as the benchmark. $Y(a)$ is the wealth at age a . The optimal value of (1), $V(a)$, is the QALY of the remaining lifetime at age a .

One obtains the first-order condition of problem (1),

$$s(t)u'(c_t) = s(a)u'(c_a)e^{\theta(t)-\theta(a)-r(t-a)}, \quad \forall t \in (a, T) \quad (2)$$

The subscript of t or a denotes the time period or the age. Note that Eq 2 characterizes the condition of optimal inter-temporal consumption allocation, which may also be derived intuitively as follows. Fix hypothetically all the optimal consumption levels over the lifetime except two points in time, a and t . It follows from the lifetime budget constraint in (1) that spending one more dollar at time a implies spending $e^{r(t-a)}$ dollars less at time t . The extra utility derived from the last one dollar spent on consumption at time a , i.e. $e^{-\theta(a)}u'(c_a)$, should be equal to the extra expected utility if one instead saves that one

⁴ Theoretically, while discounting future *consumption* at the market rate of interest may be reasonable, discounting future *utility* at this rate is questionable, especially since the probability of survival has already been accounted for. Assuming expected utility maximization, we adopt no further discount on future utility. If desired, a *constant* discount rate (as required by inter-temporal consistency) can be added with only notational and computational complications. Our $\theta'(t)$ may then be taken as the rate of depreciation (taken as zero by some other analysts) plus the pure rate of time discount.

dollar and then consumes the interest-augmented savings at any later time t , namely, $e^{-\theta(a)}u'(c_a) = e^{-\theta(t)}u'(c_t)e^{r(t-a)}s_a(t)$; hence Eq 2.

2. Analysis

Before characterizing the age-dependent VSL, we need to examine how the marginal utility value of income, i.e. the marginal contribution of income at any age to QALY, $\partial V(a)/\partial Y(a)$, may change over the life cycle.⁵ Re-write problem (1) into a standard Lagrangean form,

$$V(a) = \text{Max}_a \left\{ \int_a^T u(c_t) s_a(t) e^{-[\theta(t)-\theta(a)]} dt + \lambda(a) \left[Y(a) - \int_a^T c_t e^{-r(t-a)} dt \right] \right\} \quad (3)$$

where the multiplier $\lambda(a)$ is the utility value of income at age a . It follows from the first-order condition of the above problem that, $u'(c_t) s_a(t) e^{-[\theta(t)-\theta(a)]} = \lambda(a) e^{-r(t-a)}$ and $\partial V(a)/\partial Y(a) = \lambda(a)$, from which, in light of Eq 2, one readily obtains the following, a known result in the literature due to Johansson 2002 (p. 257). We define $\gamma(a) \equiv r - \theta'(a) - \delta(a)$.

Lemma 1. At any age, the (marginal) utility value of income declines at the rate of interest minus the sum of time preference rate and the mortality rate, both at that age. i.e. $\forall a \in (0, T), \lambda'(a) = [-\gamma(a)]\lambda(a)$.

Since having one more dollar at age a is equivalent to having instead one more dollar plus the interest return later on, the marginal effect of having one extra dollar at age a on the

⁵ It should be carefully noted that the so called utility value of income is the effect on the remaining QALY of an extra dollar of income at an age, while marginal utility of consumption is, of course, instantaneous. We deliberately use the term of “(marginal) utility value of income” to highlight the lasting effect of income at any age upon QALY of the remaining life time.

utility value over the rest of one's life must decline at the market interest rate over an optimal path, ignoring time preference and mortality. On the other hand, such a marginal effect in utility value has to be adjusted by the time preference parameter and the mortality rate as well, hence Lemma 1.

As analyzed earlier, VSL at age a , denoted as $D(a)$, equals $V(a)/\lambda(a)$. As already shown in Johansson 2002 (p. 257 and Endnote 9 on p. 262), Eq 2 and Lemma 1 imply that,

$$D'(a) = \frac{V'(a)}{\lambda(a)} + \gamma(a) \frac{V(a)}{\lambda(a)} = \frac{rV(a) - u(c_a)}{\lambda(a)} \quad (4)$$

It remains unknown, however, how VSL may change as one ages in a way that is *explicitly* informed by the difference between the market interest rate and the discounting rates of time preference and mortality, $\gamma(a)$. In other words, one stops halfway in analyzing the effects of the said parameters on age-related VSL if not proceeding further to study the way in which the instantaneous utility $u(c_a)$ is affected by the underlying factors. We now introduce.

Proposition 1. For any age a , the instantaneous consumption and utility change in such a way,

$$\frac{1}{c_a} \cdot \frac{dc_a}{da} = [-\gamma(a)] / \varepsilon_{u'c} \quad (5)$$

$$\frac{1}{u(c_a)} \cdot \frac{du(c_a)}{da} = [-\gamma(a)] \frac{\varepsilon_{uc}}{\varepsilon_{u'c}} \quad (6)$$

where $\varepsilon_{u'c}$ is the consumption elasticity of marginal utility and ε_{uc} the consumption elasticity of utility (when the consumption equals c_a)

Proof: See the appendix.

Eqs 5 and 6 are both economically intuitive. Eq 5 simply restates the FOC, Eq 2. At optimum, the percentage change in marginal utility as caused by an one-percent change in

consumption, i.e. $(\frac{1}{c_a} \cdot \frac{dc_a}{da}) \cdot \varepsilon_{u'c}$, equals the percentage rate of change in marginal utility along the optimal consumption path, which by Eq 2, is $\frac{1}{u'(c_a)} \cdot \frac{du'(c_a)}{da} = \theta'(a) + \tau(a) - r = -\gamma(a)$. Note the percentage rate of change in marginal utility along the optimal consumption path as such has incorporated the effects of the market interest as well as time preference and mortality. As to Eq 6, $[\frac{1}{u(c_a)} \cdot \frac{du(c_a)}{da}] / \varepsilon_{uc}$ amounts to the percentage rate of change in optimal consumption at age a, hence, by Eq 5, equaling $[-\gamma(a)] / \varepsilon_{u'c}$.

Apart from the intuitive interpretation of Eq 5 as given above, Eq 6 may be viewed alternatively as follows. The change in consumption affects utility through the marginal utility of consumption (u'), or in proportionate terms, through the consumption elasticity of utility (ε_{uc}). Thus, the proportionate change in utility as one ages $[\frac{1}{u(c_a)} \cdot \frac{du(c_a)}{da}]$ is proportional to the product of γ and ε_{uc} . But it is inversely proportional to the proportionate decrease in the marginal utility of consumption as consumption increases ($-\varepsilon_{u'c}$). To see the intuitive meaning of this, consider the case where the excess γ is positive (the reverse applies if this is negative). As the rate of interest dominates mortality and depreciation/time preference combined, it is optimal to consume more in the future, making u increases with age. However, if an increase in consumption decreases its marginal utility rapidly, the scope for this increase in utility is small.

For the special case where the effect of the market interest rate and that of the discounting factors (time preference and mortality) cancel out ($\gamma = 0$), the optimal amount of consumption remains constant as one ages, and hence $u(c_a)$ also remains constant. On the other hand, Johansson (2002) has shown that “VSL is independent of age in the case where optimal consumption is constant across the entire life cycle” (p.258). Is the above appealing intuition really only an illusion?

The key to the above seemingly inconsistency lies in the fact that an infinite lifespan is assumed in Johansson's formulation (as well as in some other studies, e.g. Rosen 1988). This assumption, though making technical treatment much easier than otherwise, is certainly unrealistic and responsible for Johansson's (2002) counter-intuitive result. One has to turn to a framework with a limited maximal lifespan as above to address the age-profile of VSL, particularly in the case where optimal consumption remains constant throughout.

As it turns out, within such a lifecycle framework with a finite maximal possible lifespan, even in the case of a constant optimal amount of consumption over the whole lifetime, VSL does decline as one ages. More strikingly, it declines with age at an accelerating rate.

Proposition 2. If $\theta'(t) + \delta(t) = r$ for any age t over the lifetime, then VSL monotonically declines *at an increasing speed* as one ages.

Proof: See the appendix.

Since the market interest rate and the discounting factors (the rates of time preference and mortality) net out, the marginal utility value of income, $\lambda(a)$, remains constant by Lemma 1. The optimal amount of consumption also remains unchanged, and therefore the expected QALY, $V(a)$, which is updated after surviving each age essentially in a Bayesian manner, not only decreases but decreases at an increasing rate as one ages, as shown as follows. Under the precondition of Proposition 2, the remaining QALY

$$V(a) = u(c_a) \int_a^T s_a(t) e^{-[\theta(t) - \theta(a)]} dt = u(c_a) \int_a^T e^{-r(t-a)} dt = \frac{u(c_a)}{r} [1 - e^{-r(T-a)}] \text{ not only declines}$$

but does so at an accelerating rate of the order of $e^{ra} \cdot e^{-rT}$ (note the instantaneous utility

$u(c_a)$ remains constant). As a consequence, the age-dependent VSL, equaling $V(a)/\lambda(a)$, not only declines but does so at an increasing rate with age. Incidentally, an increase in the maximal lifespan T helps to reduce the speed of decline in VSL. Thanks to progress in medical sciences against various diseases and effective public health policies, longevity has steadily and significantly improved in the past century, especially after WWII. This may render less noticeable the decline in VSL, particularly for those ages that used to be viewed as high ages but now are taken as (relatively) young.

Alternatively, we obtain from Eq 4 that for declining VSL, the speed at which VSL declines is equal to $|D'(a)| = \frac{u(c_a) - rV(a)}{\lambda(a)}$, which increases with age since $u(c_a)$ and $\lambda(a)$ both remain constant but $V(a)$ declines as shown in the above (details are found in the appendix).

It is worth emphasizing that it is the limitation of the maximal remaining lifetime, as explained already in the above, that drives the decline of VSL with age. Moreover, it also drives the acceleration of such a decline.

Realistically, the assumption adopted in Proposition 2 that “ $\theta'(t) + \delta(t) = r$ for any age t over the (whole) lifetime” cannot hold, since $s_t(T) = e^{-\int_t^T \delta(\tau) d\tau} = 0$ requires that the mortality rate $\delta(\tau)$ be infinitely large for at least a certain interval along the time horizon, implying $-\theta'(t)$ is infinitely large during the said period of time/ages. But the situation in which the capability to derive pleasure from consumption becomes infinitely high (or, alternatively interpreted, the preference depreciation rate θ' becomes infinitely negative) stands as an absurdity. A less implausible scenario is that $\theta'(t) + \delta(t) = r$ for any age t over almost all lifetime, but a small range of ages just below the maximal lifespan T . If such a range is sufficiently small at least from the perspective of a young person, during which the capability to derive pleasure from consumption is not infinitely high, then VSL declines with age still.

A much more plausible scenario is that there exists a certain age T' , after which the sum of the mortality rate and depreciation rate exceeds the market interest rate, and before which it is more or less stable and close to the market interest rate. For the sake of illustration, we may simply assume that $\theta'(t) + \delta(t) = r$ for any $t \leq T'$ and $\theta'(t) + \delta(t) > r$ for any $t > T'$. Under such a simplification, we have

Proposition 3. If there exists a certain age T' such that $\theta'(t) + \delta(t) = r$ for any $t \leq T'$ and $\theta'(t) + \delta(t) > r$ for any $t > T'$, then VSL monotonically declines with age.

Proof: See the appendix.

As has been analyzed earlier, that the netting out of the market interest rate and the discounting factors (the time preference and mortality) implies a constant marginal utility value of income, $\lambda(a)$. The optimal amount of consumption also remains unchanged before one reaches age T' but declines thereafter. VSL thus declines with age. For an age above T' , the utility value of income at each age declines but QALY declines even faster, rendering a decline in VSL. It must be pointed out that Proposition 3 does not rely on finiteness of the maximal possible lifespan T . Even for $T = \infty$, it remains true. Moreover, it is evident from the above argument and Proposition 2 that the condition of “ $\theta'(t) + \delta(t) > r$ for any $t > T'$ ” in Proposition 3 can be relaxed as “ $\theta'(t) + \delta(t) \geq r$ for any $t > T'$ ” (slight modification of the algebra in the proof of Proposition 3 in the appendix suffices to demonstrate a declining VSL profile).

3. Discussion

Our Proposition 1 characterizes the changes in optimal consumption and instantaneous (pre-discounted/depreciated) utility as one ages, stating that such changes are proportionate to the difference between the market interest rate and the sum of the time

discount rate and the mortality rate. Consumption elasticity of (marginal) utility determines the extent to which changes in optimal consumption and instantaneous utility are effected by the market interest and the discounting factors. This technical novelty enables us to show, as summarized in Propositions 2 and 3, that VSL declines if the market interest rate exceeds or equals the sum of age-specific depreciation rate and the mortality rate at any age in the remaining lifetime. However, such a relationship between these three parameters does not seem to be applicable to all ages across the life cycle. For instance, in conducting their iterative simulation on the effect of life protection on the lifetime expectancy, Ehrlich and Yin (2005) use data of long-term returns on a portfolio of financial and non-financial assets in the US during half a century after WWII to obtain a conservative estimate of the market interest rate as 3.2% and indirectly estimate the discount rate as 1.6% from statistical data of family wealth. The age-specific mortality rates are calibrated from the *Vita and Health Statistics of the US*, being less than the difference between the interest rate and the discount rate (1.6%) up to a fairly high age (refer to Table 1 on p.137 of Ehrlich and Yin 2005). In other words, $r > \theta'(t) + \delta(t)$ holds for many young ages. The mortality rate, statistically speaking, increases steeply only after a certain high age.⁶

Thus, realistically, against the market interest rate, both the time preference and the mortality rate change slowly for some young ages and then mortality (perhaps the rate of depreciation in the capacity to derive utility from consumption too) grows quickly for high ages especially when approaching the end of the lifespan, i.e. $\theta'(t) + \delta(t) < r$ for young ages t , but $\theta'(t) + \delta(t) \geq r$ for high ages. Our above analysis then predicts that under this

⁶ It may be noted that similar patterns of mortality – the mortality rates remain fairly low and stable up to one’s sixties and then increase rather sharply at ages approaching the maximum lifespan – are also found for Australians (Trewin 2002, especially Appendix 2, Death Rates), and quite likely for many, if not almost all, other peoples too.

most plausible circumstance, VSL exhibits an inverted-U shape. The reason is as follows. For high ages as such, by Proposition 3, VSL declines. But for sufficiently young ages a , by Proposition 1, the instantaneous utility increases with age. The mortality is almost negligible against the market interest rate and the time discounting rate is considerably

lower than the market interest rate, rendering $\int_a^T [u(c_t)/u(c_a)]s_a(t)e^{-[\theta(t)-\theta(a)]}dt$ a

considerably large value relative to one at sufficiently young ages. By Eq 5,

$$D'(a) = \frac{r[V(a)/u(c_a)] - 1}{[\lambda(a)/u(c_a)]} > 0.$$

It may be noted that we separately deal with the depreciation rate and the mortality rate as two interdependent factors in our formulation for the sake of neatness of analysis. That they are often positively correlated especially when quite old can only make our analysis more appealing.

One important issue to understanding the age dependency of VSL, which has remained unaddressed yet in the literature, is protection against mortality hazard. Life protection is incorporated in Ehrlich and Yin's (2005) numerical simulation in that investment on life protection reduces the hazard rates. Perhaps more than that, investment on health maintenance and improvement not only produces influence on the longevity but also significantly affects the well beings *per se*, hence affecting QALY through two channels – improving in both quantity and quality. Then, one crucial question to be asked is: when and how much one may optimally invest on health and, consequently, how the age-profile of VSL may thus be affected? Future study along this line might prove to be (technically) challenging and fruitful.

Appendix

The proof of Proposition 1. To understand how the instantaneous utility changes as one ages, we first come to grips with how the instantaneous consumption is affected by the market interest rates as well as the discounting factors. Rewrite Eq 2 as

$$s(a)u'(c_a)e^{ra-\theta(a)} = u'(c_0), \text{ of which differentiation w.r.t. age } a \text{ yields } u''(c_a)\frac{dc_a}{da} =$$

$-\gamma(a)u'(c_a)$. Hence, $\frac{1}{c_a} \cdot \frac{dc_a}{da} = [-\gamma(a)]/\varepsilon_{u'c}$, where $\varepsilon_{u'c} \equiv (\partial u'/\partial c)c/u'$ is the consumption elasticity of marginal utility (when the consumption equals c_a).

$$\text{Consequently, } \frac{du(c_a)}{da} = u'(c_a) \cdot \frac{dc_a}{da} = \frac{\gamma(a)}{-u''(a)}(u'(a))^2. \text{ Hence, } \frac{1}{u(c_a)} \cdot \frac{du(c_a)}{da}$$

$= [-\gamma(a)] \frac{\varepsilon_{uc}}{\varepsilon_{u'c}}$ where $\varepsilon_{uc} \equiv (\partial u/\partial c)c/u$ is the consumption elasticity of utility (when the consumption equals c_a). **QED**

The proof of Proposition 2. By Eq 4, the change rate of VSL $D'(a) = \frac{rV(a) - u(c_a)}{\lambda(a)}$.

Under the assumption of the market interest rate equaling the sum of the discounting factors of time preference and mortality, by Proposition 1, the instantaneous consumption and the instantaneous utility (before being discounted by time preference and mortality)

$$\text{remain constant, hence } D'(a) = \frac{1}{\lambda(a)} \left\{ r \int_a^T u(c_a) e^{-\int_a^t [\theta'(s) + \delta(s)] ds} dt - u(c_a) \right\}$$

$$= \frac{1}{\lambda(a)} \left[r \int_a^T u(c_a) e^{-r(t-a)} dt - u(c_a) \right] = -\frac{e^{-r(T-a)}}{\lambda(a)} u(c_a) < 0. \text{ That is, VSL declines at a rate of}$$

$e^{ra} K(T, a)$, where $K(T, a) \equiv \frac{e^{-rT}}{\lambda(a)} u(c_a)$, is independent of the age a , under the condition

$\gamma(a) = 0$ for any age a . **QED**

The proof of Proposition 3. We first consider any age $a < T'$. By Proposition 1, $\theta'(t) + \delta(t) = r$ for any $t < T'$ implies that $u(c_t)$ remains constant for $t < T'$. Thus, by Eq

$$4, D'(a) = \frac{rV(a) - u(c_a)}{\lambda(a)}$$

$$= \frac{1}{\lambda(a)} \left\{ r \left[\int_a^{T'} u(c_{T'}) s_a(t) e^{-[\theta(t) - \theta(a)]} dt + \int_{T'}^T u(c_t) s_a(t) e^{-[\theta(t) - \theta(T')] - [\theta(T') - \theta(a)]} dt \right] - u(c_{T'}) \right\}$$

$$\begin{aligned}
&= \frac{1}{\lambda(a)} \left\{ r \left[\int_a^{T'} u(c_{T'}) e^{-\int_a^t [\theta'(s) + \delta(s)] ds} dt + \int_{T'}^T u(c_t) e^{-\int_a^{T'} [\theta'(s) + \delta(s)] ds} e^{-\int_{T'}^t [\theta'(s) + \delta(s)] ds} dt \right] - u(c_{T'}) \right\} \\
&= \frac{1}{\lambda(a)} \left\{ r \left[\int_a^{T'} u(c_{T'}) e^{-r(t-a)} dt + \int_{T'}^T u(c_t) e^{-r(T'-a)} e^{-\int_{T'}^t [\theta'(s) + \delta(s)] ds} dt \right] - u(c_{T'}) \right\} \\
&= \frac{e^{-r(T'-a)}}{\lambda(a)} \left\{ \int_{T'}^T r u(c_t) e^{-\int_{T'}^t [\theta'(s) + \delta(s)] ds} dt - u(c_{T'}) \right\}
\end{aligned}$$

Note $\theta'(t) + \delta(t) > r$ for any $t > T'$, again by Proposition 1, $u(c_t)$ declines at any age

$$\begin{aligned}
t > T'. \text{ Thus, for any } a < T', D'(a) &< \frac{e^{-r(T'-a)}}{\lambda(a)} \left\{ u(c_{T'}) \int_{T'}^T r e^{-\int_{T'}^t [\theta'(s) + \delta(s)] ds} dt - u(c_{T'}) \right\} \\
&< \frac{e^{-r(T'-a)}}{\lambda(a)} u(c_{T'}) \left[\int_{T'}^T r e^{-r(t-T')} dt - 1 \right] = -\frac{e^{-r(T-a)}}{\lambda(a)} u(c_{T'}) < 0.
\end{aligned}$$

We now turn to any age a above T' . Since $\theta'(a) + \delta(a) > r$, the instantaneous utility declines with ages as is analyzed already. Consequently,

$$\begin{aligned}
D'(a) &= \frac{1}{\lambda(a)} \left\{ r \int_a^T u(c_t) s_a(t) e^{-[\theta(t) - \theta(a)]} dt \right\} - u(c_a) < \frac{1}{\lambda(a)} u(c_a) \left\{ \int_a^T r e^{-\int_a^t [\theta'(s) + \delta(s)] ds} dt - 1 \right\} \\
&< \frac{1}{\lambda(a)} u(c_a) \left[\int_a^T r e^{-r(t-a)} dt - 1 \right] = -\frac{e^{-r(T-a)}}{\lambda(a)} u(c_a) < 0. \qquad \text{QED}
\end{aligned}$$

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