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A General Method to Create Lorenz Models

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Abstract:
There are currently about two dozen Lorenz models available in the literature for fitting grouped income distribution data. A general method to construct parametric Lorenz models of the weighted product form is offered in this paper. First, a general result to describe the conditions for the weighted product model to be a Lorenz curve, created by using several component parametric Lorenz models, is given. We show that the key property for an ideal component model is that the ratio between its second derivative and its first derivative is increasing. Then, a set of Lorenz models, consisting of a basic group of models along with their convex combinations, is proposed, and it is shown that any model in the set possesses this key property. Equipped with this general result and the model set, we can create a range of different weighted product Lorenz models. Finally, test results are presented which demonstrate that there may be many satisfactory models among those created. The proposed method can be generalized by finding other models with this key property.

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1. Introduction

The parametric Lorenz model is an important tool in income distribution analysis. Many researchers have contributed to the literature on Lorenz models. Normally, each contribution provides an individual model with test results applied to some empirical data. Schader and Schmid (1994) give an exhaustive list of the models until the mid-1990s. More recent models include those proposed by Ogwang and Rao (1996, 2000), Ryn and Slottje (1996) and Sarabia *et al.* (1999, 2001). Overall, there have been about two dozen Lorenz models proposed in the literature to this point. For a comparison of existing models, see Cheong (2002) and Schader and Schmid (1994).

The shortcomings of existing models in the literature include the following. First, they fail to explain why a specific functional form can be used to model income data for a variety of sources (Ryn & Slottje 1996). Second, some models do not give a global approximation to the actual data. Specifically, they may fit the data well at some parts of the distribution, but are poor fits elsewhere (Basmann *et al*. 1990; Ryn & Slottje 1996; Ogwang & Rao 2000). Third, some models do not satisfy the definition of the Lorenz curve (Ortega *et al*. 1991; Schader & Schmid 1994). We address these limitations by providing a general method to construct Lorenz models.

The general method we propose entails constructing weighted product models by using a special set of parametric Lorenz models. The simplest weighted product model is the multiplicative form of two component Lorenz models. We first provide general conditions for this simplest form to satisfy the definition of the Lorenz curve and find that an ideal component for the multiplicative form is that the ratio between its second derivative and its first derivative is increasing. Equipped with this result, we provide a general theorem which sets forth the conditions for a weighted product model of finite Lorenz models to satisfy the definition of the Lorenz curve. We then suggest a special set X of parametric Lorenz models with this ideal property. The set X consists of a few simple Lorenz models as well as their convex combinations. These simple models can be understood as generalizations of the Lorenz curve associated with the classical Pareto distribution. With the aid of the general theorem, and the set X, we can create a series of weighted product models. We find that a significant feature of some of the models so created is that they can bend into angles, which some of the traditional models cannot do. Ogwang & Rao (2000) study the multiplicative form of two Lorenz models and the convex combination form of two Lorenz models in their hybrid models, but do not discuss the general condition for the former form to be a Lorenz curve. Instead, they find that some Lorenz models of the former variety are inferior to the latter following empirical testing.

The structure of the paper is as follows. Sufficient conditions for the weighted product model to satisfy the definition of the Lorenz curve are set out in the next section. The basic group of Lorenz models is proposed in Section 3. The special set X of parametric Lorenz models is provided in Section 4. Some selected examples of the weighted product models created from X are suggested in Section 5, while the test results from these models are reported in Section 6. Some concluding remarks and suggestions for further research are offered in the final section.

2. The general method to create Lorenz models

We call $L(p)$ a Lorenz curve, if $L(p)$ is defined on $[0,1]$, possesses a continuous third derivative and satisfies the conditions that $L(0)=0$, $L(1)=1$, $L'(p)\geq 0$ and $L''(p)\geq 0$. To commence with, consider the function of the multiplicative form:

$$\tilde{L}(p)=f(p)^{\alpha}g(p)^{\eta}, \quad \alpha\geq 0 \ \text{ and } \ \eta\geq 0$$

where both the component function $f(p)$ and $g(p)$ are parametric Lorenz curves. It follows that $\tilde{L}(p)$ is a Lorenz curve if $\tilde{L}'(p)\geq 0$ and $\tilde{L}''(p)\geq 0$. But

$$\tilde{L}'(p)=\alpha f(p)^{\alpha-1}f'(p)g(p)^{\eta}+\eta g(p)^{\eta-1}g'(p)f(p)^{\alpha}\geq 0$$

is true, therefore, we only have to consider the condition for $\tilde{L}''(p)\geq 0$. Since

$$\begin{aligned}\tilde{L}''(p)=&\ \alpha(\alpha-1)f(p)^{\alpha-2}f'(p)^{2}g(p)^{\eta}\\ &+\alpha f(p)^{\alpha-1}f''(p)g(p)^{\eta}+\alpha\eta f(p)^{\alpha-1}f'(p)g(p)^{\eta-1}g'(p)\\ &+\eta(\eta-1)g(p)^{\eta-2}g'(p)^{2}f(p)^{\alpha}\\ &+\eta g(p)^{\eta-1}g''(p)f(p)^{\alpha}+\eta\alpha g(p)^{\eta-1}g'(p)f(p)^{\alpha-1}f'(p),\end{aligned}$$

it follows $\tilde{L}''(p)\geq 0$ if both $\alpha\geq 1$ and $\eta\geq 1$, as noted by Ogwang & Rao (2000).

We can consider other cases. Denote the sum of the first three terms on the right-hand side of the above equation as $h(p)$ and the sum of the remaining three terms as

$t(p)$. Thus, we need only find the condition for both $h(p) \ge 0$ and $t(p) \ge 0$. Since

$$\text{(1)} \qquad \frac{h(p)}{\alpha f(p)^{\alpha-2} g(p)^{\eta-1}} = (\alpha-1) f'(p)^2 g(p) \\ + f(p) f''(p) g(p) + \eta f(p) f'(p) g'(p),$$

we can conclude that $h(p) \ge 0$ if $\alpha \ge 1/2$, $\eta \ge 0$, $\alpha + \eta \ge 1$ and $f'''(p) \ge 0$, since if we write the right-hand side of this equation as $\psi(p)$, we find that $\psi(0) = 0$ and $\psi'(p) \ge 0$ for any $p \in [0,1]$. Moreover, assume $\alpha \ge 0$, $\eta \ge 0$, $\alpha + \eta \ge 1$ and $f_1(p) = f''(p)/f'(p)$ is increasing. Rewrite the right-hand side of (1) as

$$\left[(\alpha-1) f'(p) g(p) + f(p) f_1(p) g(p) + \eta f(p) g'(p)\right] f'(p).$$

Let the function between the braces be $\varphi(p)$, we can verify that $\varphi(0) = 0$ and $\varphi'(p) \ge 0$ for any $p \in [0,1]$. Consequently, we can again conclude that $h(p) \ge 0$.

Note further

$$\frac{t(p)}{\eta g(p)^{\eta-2} f(p)^{\alpha-1}} = (\eta-1) g'(p)^2 f(p) \\ + g(p) g''(p) f(p) + \alpha g(p) g'(p) f'(p).$$

The right-hand side of this equation is exactly the same as that of Equation (1), if we exchange the position of $g(p)$ and $f(p)$, and the position of α and η. Thus we have $t(p) \ge 0$ if $\alpha \ge 0$, $\eta \ge 1/2$, $\alpha + \eta \ge 1$ and $g'''(p) \ge 0$. Furthermore, we also have $t(p) \ge 0$ if $\alpha \ge 0$, $\eta \ge 0$, $\alpha + \eta \ge 1$ and $g_1(p) = g''(p)/g'(p)$ is increasing.

To synthesize the discussion, we have the following lemma:

Lemma 1. Assume both $f(p)$ and $g(p)$ are Lorenz curves. It follows $\tilde{L}(p) = f(p)^{\alpha} g(p)^{\eta}$ is a Lorenz curve if any of the following conditions holds:

i). $a \ge 1$ and $\eta \ge 1$.

ii). $\alpha \ge 1/2$, $\eta \ge 1$ and $f'''(p) \ge 0$ on $[0,1]$.

iii). $\alpha \ge 0$, $\eta \ge 1$ and $f''(p)/f'(p)$ is increasing on $[0,1]$.

iv). $\alpha \ge 1/2$, $\eta \ge 1/2$ and both $f'''(p) \ge 0$ and $g'''(p) \ge 0$ on $[0,1]$.

v). $\alpha \ge 0$, $\eta \ge 1/2$, $\alpha + \eta \ge 1$, $f''(p)/f'(p)$ is increasing and $g'''(p) \ge 0$ on $[0,1]$.

vi). $\alpha \geq 0$, $\eta \geq 0$, $\alpha + \eta \geq 1$, and both $f''(p)/f'(p)$ and $g''(p)/g'(p)$ are increasing on $[0,1]$.

By symmetry, under the assumptionsx that g''/g' is increasing and $f'''(p) \geq 0$ on $[0,1]$, statement v) of the lemma implies that $\tilde{L}(p)$ is a Lorenz curve if $\eta \geq 0$, $\alpha \geq 1/2$ and $\alpha + \eta \geq 1$. Note that the condition $\alpha + \eta \geq 1$ cannot be relaxed. If, to the contrary, $\alpha \geq 0$, $\eta \geq 0$ and $\alpha + \eta < 1$, then by letting $f(p) = g(p) = p$, we get $\tilde{L}(p) = p^{\alpha+\eta}$, which is not a Lorenz curve. According to Lemma 1, the stricter the condition imposed upon a component function, the larger is the admissible range of the corresponding exponential parameter. For a pair of fixed component Lorenz curves $f(p)$ and $g(p)$, the ideal situation is that both f''/f' and g''/g' are increasing. Statement vi) then asserts that the admissible range of α and η is

$$\{(\alpha, \eta) \mid \alpha \geq 0, \eta \geq 0, \alpha + \eta \geq 1\},$$

which achieves a state of maximum. An important special case of the multiplicative model of two component Lorenz models is studied by Sarabia *et al*. (1999); namely,

$$L_S(p) = p^{\alpha} L(p)^{\eta},$$

and we then have the following result by Lemma 1:

Corollary. Assume $L(p)$ is a Lorenz curve. Then $L_S(p)$ is a Lorenz curve if any one or more of the following conditions holds:
i). $\alpha \geq 0$ and $\eta \geq 1$;
ii). $\alpha \geq 0$, $\eta \geq 1/2$, $\alpha + \eta \geq 1$ and $L'''(p) \geq 0$;
iii). $\alpha \geq 0$, $\eta \geq 0$, $\alpha + \eta \geq 1$ and $L''(p)/L'(p) \geq 0$.

Sarabia *et al*. (1999) provide statement i) of the corollary, but they impose the condition $L'''(p) \geq 0$. The first two statements are also provided, and elaborated on, in Wang, Ng & Smyth (2007) (hereafter WNS, 2007). Let

$$X_0 = \{L(p) \mid L(p) \text{ be a Lorenz curve with increasing } L''(p)/L'(p)\}.$$

Consider a series of component Lorenz models $\{L_i(p)\}_{i=1}^{m} \subset X_0$. Denote the weighted product model

$$\tilde{L}(p) = L_1(p)^{\alpha_1} L_2(p)^{\alpha_2} \cdots L_m(p)^{\alpha_m}, \quad \alpha_1 \ge 0, \alpha_2 \ge 0, \cdots, \alpha_m \ge 0 .$$

Furthermore, let

$$Y_0 = \{L(p) \mid L(p) \text{ be a Lorenz curve with } L'''(p) \ge 0\},$$

$$Z_0 = \{L(p) \mid L(p) \text{ be a Lorenz curve}\}.$$

Therefore, Z_0 contains all possible parametric Lorenz curves. We have $X_0 \subset Y_0 \subset Z_0$. Our general method of creating Lorenz models is described in the following theorem, which follows from statements iii), v) and vi) of Lemma 1:

Theorem 1. Let $L(p) \in Z_0$. Then $\tilde{L}(p)L(p)^{\eta}$ is a Lorenz curve if $\eta \ge 1$. Let $L(p) \in Y_0$ and assume that there exists an exponent, say, $\alpha_i \in \{\alpha_1, \cdots, \alpha_m\}$, such that $\alpha_i + \eta \ge 1$. Then $\tilde{L}(p)L(p)^{\eta}$ is a Lorenz curve if $\eta \ge 1/2$. Let $\{L_i(p)\}_{i=1}^{m} \subset X_0$ and assume that there exists a pair of exponents within $\{\alpha_1, \cdots, \alpha_m\}$, say, α_i and α_j with $\alpha_i + \alpha_j \ge 1$. Then $\tilde{L}(p)$ itself is a Lorenz curve.

A weighted product model can also be called a Cobb-Douglas model. Whether our general method is feasible depends on whether we can find the set, X_0. If so, we can, for example, create new Lorenz models combining $\tilde{L}(p)$ and any $L(p)$ extant in the literature, according to the first statement of Theorem 1. Unfortunately, we are not able to find the entire set, X_0. However, we can consider a less general alternative by finding a subset $X \subset X_0$ and construct weighted product models $\tilde{L}(p)$ with the elements of X as components. In the next section we suggest such a set.

3. Generalized Pareto Lorenz models

3.1. A set of generalized Pareto Lorenz models

Consider the set of basic models:

(1) $L_1(p) = p$,

(2) $L_2(p) = 1 - (1-p)^{\beta}, \quad \beta \in (0,1]$,

(3) $L_{\lambda}(p) = \dfrac{e^{\lambda p} - 1}{e^{\lambda} - 1}, \quad \lambda > 0$,

(4) $$L_3(p) = 1 - L_{\lambda_1}(1-p)^{\beta_1}, \quad \beta_1 \in (0,1], \quad \lambda_1 \in (-\infty,0) \cup (0, \ln \beta_1^{-1}],$$

(5) $$L_4(p) = 1 - \left(1 - L_{\lambda_2}(p)\right)^{\beta_2}, \quad \beta_2 \in (0,1], \quad \lambda_2 \in [\ln \beta_2, 0) \cup (0,+\infty).$$

These functions possess the derivative of any order. $L_2(p)$ is the Lorenz curve associated with the classical Pareto distribution. $L_\lambda(p)$ is the Lorenz curve suggested by Chotikapanich (1993) with λ its unique parameter. $L_\lambda(p)$ is satisfied for any $\lambda \neq 0$,

$$L_\lambda(p) \geq 0 \text{ and } L'_\lambda(p) \geq 0 \text{ on } [0,1],$$

$$L_\lambda^{(n)}(p) = \lambda^{n-1} L_\lambda^{(n-1)}(p), n = 2,3,\cdots.$$

To avoid confusion in the ensuing discussion, note that unlike the parameter λ in the model $L_\lambda(p)$ or $L_\lambda(1-p) = (e^\lambda - 1)^{-1}(e^{\lambda(1-p)} - 1)$, the symbol i in $L_i(p)$ does not represent a parameter of the model. $L_1(p)$ is a special case of $L_2(p)$ because it can be obtained by letting $\beta = 1$ in the latter. $L_\lambda(p)$ is equal to $L_1(p)$ when $\lambda \to 0$. $L_3(p)$ is the Lorenz model provided by Wang & Smyth (2007) (hereafter WS, 2007) and is a generalization of $L_2(p)$. $L_4(p)$ is a new model and is also a generalization of $L_2(p)$. We call these basic models generalized Pareto (GP) models.

Note first that we have the following two inequalities

(6) $$(1-\beta_1)L'_{\lambda_1}(1-p) - \lambda_1 L_{\lambda_1}(1-p) \geq 0,$$

(7) $$(1-\beta_2)L'_{\lambda_2}(p) + \lambda_2\left(1 - L_{\lambda_2}(p)\right) \geq 0,$$

with the parameters defined in (4) and (5) respectively. By the definition of $L_\lambda(p)$, the inequality (6) is equivalent to

$$\frac{\lambda_1}{e^{\lambda_1} - 1}\left(1 - \beta_1 e^{\lambda_1(1-p)}\right) \geq 0.$$

This inequality holds because $\lambda_1(e^{\lambda_1} - 1)^{-1} \geq 0$ for any $\lambda_1 \neq 0$ and $1 - \beta_1 e^{\lambda_1(1-p)} \geq 0$ if β_1 and λ_1 are defined by (4). Meanwhile, (7) is equivalent to

$$\frac{\lambda_2}{e^{\lambda_2} - 1}\left[e^{\lambda_2} - \beta_2 e^{\lambda_2 p}\right] \geq 0.$$

It also holds if β_2 and λ_2 are defined by (5).

$L_4(p)$ is also a Lorenz curve and we have only to verify both $L'_4(p) \geq 0$ and

$L_4''(p) \geq 0$. However,

$$L_4'(p) = \beta_2 \left(1 - L_{\lambda_2}(p)\right)^{\beta_2 - 1} L_{\lambda_2}'(p).$$

Thus $L_4'(p) \geq 0$. Moreover,

$$L_4''(p) = \beta_2 \left(1 - L_{\lambda_2}(p)\right)^{\beta_2 - 2} \left[(1 - \beta_2) L_{\lambda_2}'(p) + \lambda_2 \left(1 - L_{\lambda_2}(p)\right)\right] L_{\lambda_2}'(p).$$

Therefore, $L_4''(p) \geq 0$ by (7).

Furthermore, we have

Lemma 2. Every GP model $L(p)$ is a Lorenz curve with increasing $L''(p)/L'(p)$.

The statement is evident for $L_1(p)$, $L_\lambda(p)$ and $L_2(p)$. Denote

$$h(p) = \frac{L_3''(p)}{L_3'(p)} = \frac{(1 - \beta_1) L_{\lambda_1}'(1 - p) - \lambda_1 L_{\lambda_1}(1 - p)}{L_{\lambda_1}(1 - p)}.$$

Thus, we have

$$h'(p) = \frac{(1 - \beta_1) L_{\lambda_1}'(1 - p)}{L_{\lambda_1}(1 - p)^2} \left[L_{\lambda_1}'(1 - p) - \lambda_1 L_{\lambda_1}(1 - p)\right].$$

The right-hand side is non-negative by (6). Therefore $L_3''(p)/L_3'(p)$ is increasing.

Denote

$$s(p) = \frac{L_4''(p)}{L_4'(p)} = \frac{(1 - \beta_2) L_{\lambda_2}'(p) + \lambda_2 \left(1 - L_{\lambda_2}(p)\right)}{1 - L_{\lambda_2}(p)}.$$

Therefore, we have

$$s'(p) = \frac{L_{\lambda_2}'(p)(1 - \beta_2)}{(1 - L_{\lambda_2}(p))^2} \left\{L_{\lambda_2}'(p) + \lambda_2 \left(1 - L_{\lambda_2}(p)\right)\right\} \geq 0,$$

which is true by (7). Therefore $L_4''(p)/L_4'(p)$ is also increasing.

Employing only the GP models and the third statement of Theorem 1, we can create many weighted product models. For example, all the following are Lorenz curves:

(8) $\left(1 - (1 - p)^\beta\right)^\eta$, $\beta \in (0,1]$, $\eta \geq 1$,

(9) $p^\alpha \left[1 - (1 - p)^\beta\right]^\eta$, $\beta \in (0,1]$,

(10) $p^{\alpha} L_{\lambda}(p)^{\eta}$, $\lambda > 0$,

(11) $p^{\alpha}\left[1 - L_{\lambda}(1-p)^{\beta}\right]^{\eta}$, $\beta \in (0,1]$, $\lambda \in (-\infty,0) \cup (0, \ln \beta^{-1}]$,

(12) $p^{\alpha}\left[1 - (1 - L_{\lambda}(p))^{\beta}\right]^{\eta}$, $\beta \in (0,1]$, $\lambda \in [\ln \beta, 0) \cup (0, +\infty)$,

where $\alpha \geq 0$, $\eta \geq 0$ and $\alpha + \eta \geq 1$ for the models specified in (9)-(12); (8) is the model provided by Rasche *et al*. (1980); (9) and (10) are models proposed by Sarabia *et al*. (1999, 2001) respectively, but with $\eta \geq 1$ imposed and (11) is suggested by WS (2007), but with $\eta \geq 1/2$ imposed. Since $L_{\lambda}(x) \to x$ when $\lambda \to 0$, (11) includes (8)-(9) as special cases and (12) includes (8)-(10) as special cases.

Assume again that $\alpha \geq 0$, $\eta \geq 0$ and $\alpha + \eta \geq 1$. By Theorem 1, a more sophisticated Lorenz model is as follows:

(13) $p^{\alpha}\left[1 - L_{\lambda_1}(1-p)^{\beta_1}\right]^{\alpha_1}\left[1 - (1 - L_{\lambda_2}(p))^{\beta_2}\right]^{\eta}$,

where $\alpha_1 \geq 0$ should be imposed. Of course, we can also impose $\alpha + \alpha_1 \geq 1$ or $\alpha_1 + \eta \geq 1$ instead. Other parameters are defined in (4)-(5). We have repeatedly avoided using a GP member in the models above. However, Theorem 1 implies that

(14) $\left(1 - (1-p)^{\beta_1}\right)^{\alpha}\left(1 - (1-p)^{\beta}\right)^{\eta}$

is also a Lorenz curve, if $\alpha \geq 0$, $\eta \geq 0$, $\alpha + \eta \geq 1$, $\beta \in (0,1]$ and $\beta_1 \in (0,1]$. Model (14) is significant. It nests

$$p^{\alpha}\left(1 - (1-p)^{\beta}\right), \quad \alpha \geq 0 \text{ and } \beta \in (0,1],$$

suggested by Ortega *et al*. (1991) and the models defined by (8)-(9). Clearly, (13) nests all other models presented here and should outperform these other models.

3.2. The convex combination of the GP models

While we have obtained a number of Lorenz models in the last section, better options still exist. Define

$$X = \{L(p) \mid L(p) \text{ is a convex combination of the GP models}\}.$$

Every element of X can be used as a Lorenz model. Note that the requirement that $h(p) = L''(p)/L'(p)$ is increasing is equivalent to $h'(p) \geq 0$ or $L'''L' - L''^2 \geq 0$ under the continuity assumption of the derivatives. This implies $L'''(p) \geq 0$ in turn, because a Lorenz curve $L(p)$ must satisfy $L'(p) \geq 0$. Let $x(p)$ and $y(p)$ be

Lorenz curves with increasing L''/L'. A sufficient condition for the weighted sum

$$L(p) = \delta x(p) + (1-\delta)y(p)$$

to have increasing L''/L', where δ is the weight coefficient and satisfies $\delta \in [0,1]$, is

(15) $$x'''y' + y'''x' - 2x''y'' \geq 0.$$

First we give a simple result.

Lemma 3. Let $X_1 \subset X_0$ be a set of Lorenz models with (15) being satisfied for any pair $x(p)$ and $y(p)$ in X_1, where X_0 is defined in Section 2 above. Then, any convex combination of the elements of X_1 belongs to X_0.

Proof. (15) implies that a convex combination of any two elements in X_1 is in X_0. Consider the convex combination of three models $x(p)$, $y(p)$ and $z(p)$ in X_1

$$L(p) = \delta_1 x(p) + \delta_2 y(p) + \delta_3 z(p),$$

where the weight coefficients satisfy that $\delta_1 \geq 0$, $\delta_2 \geq 0$ and $\delta_3 = 1 - \delta_1 - \delta_2 \geq 0$. Let $\delta = \delta_1$ and $\tilde{\delta} = \delta_2/(\delta_2 + \delta_3)$. $L(p)$ can then be rewritten as

$$L(p) = \delta x(p) + (1-\delta)\left(\tilde{\delta} y(p) + (1-\tilde{\delta})z(p)\right) = \delta x(p) + (1-\delta)u(p),$$

where $u(p) = \tilde{\delta} y(p) + (1-\tilde{\delta})z(p)$ possesses increasing L''/L' by assumption. To prove $L(p)$ is in X_0, we need only prove that (15) holds for $x(p)$ and $u(p)$. But, this result can be implied together by the following inequalities

$$x'''y' + x'y''' - 2x''y'' \geq 0,$$

$$x'''z' + x'z''' - 2x''z'' \geq 0,$$

which are true by assumption. We can verify that the convex combination of any finite elements in X_1 is in X_0 by induction.

Theorem 2. Every element of X is a Lorenz curve with increasing L''/L'.

Proof. It is obvious that every element of X is a Lorenz curve. Since every GP member possesses increasing L''/L', to complete the proof of the theorem, by Lemma 2-3, we only have to prove that any convex combination of two GP models has increasing L''/L'. Noting first that $x'''x' - x''^2 \geq 0$ implies that $\delta p + (1-\delta)x(p)$ has increasing L''/L' by (15), we only have to verify that every model of

$$L_5(p) = \delta L_2(p) + (1-\delta)L_\lambda(p),$$

$$L_6(p) = \delta L_3(p) + (1-\delta)L_\lambda(p),$$

$$L_7(p) = \delta L_3(p) + (1-\delta)L_2(p),$$

$$L_8(p) = \delta L_4(p) + (1-\delta)L_\lambda(p),$$

$$L_9(p) = \delta L_4(p) + (1-\delta)L_2(p),$$

$$L_{10}(p) = \delta L_4(p) + (1-\delta)L_3(p),$$

has increasing L''/L', where $\delta \in [0,1]$ and other admissible ranges for the parameters involved are defined in Equations (2)-(5).

Consider $L_5(p)$ first. We verify that inequality (15) holds, which is now

$$L_2'''(p)L_\lambda'(p) + L_2'(p)L_\lambda'''(p) - 2L_2''(p)L_\lambda''(p) \geq 0.$$

This inequality is equivalent to

$$(1-\beta)(2-\beta) + (1-p)^2\lambda^2 - 2\lambda(1-\beta)(1-p) \geq 0.$$

The left-hand side is equal to $\left[\lambda(1-p) - (1-\beta)\right]^2 + (1-\beta)$, which is non-negative.

Consider $L_6(p) = \delta L_3(p) + (1-\delta)L_\lambda(p)$. We prove that (15) is true, which is now equivalent to

(16) $$L_3'''(p) + \lambda^2 L_3'(p) - 2\lambda L_3''(p) \geq 0.$$

Denote the function on the left-hand of (16) as $u(p)$. It follows that:

$$\frac{u(p)}{\beta_1 L_{\lambda_1}(1-p)^{\beta_1-3}L_{\lambda_1}'(1-p)} = \left\{\left[(1-\beta_1)L_{\lambda_1}'(1-p) - \lambda_1 L_{\lambda_1}(1-p)\right] - \lambda L_{\lambda_1}(1-p)\right\}^2$$
$$+ (1-\beta_1)\left[L_{\lambda_1}'(1-p) - \lambda_1 L_{\lambda_1}(1-p)\right]L_{\lambda_1}'(1-p).$$

The right-hand side of this equation is non-negative by (6), which implies that (16) is true for any $\lambda \neq 0$.

Consider $L_7(p)$. We prove that (15) is true once more, which is equivalent to

$$L_3'''(p)(1-p)^2 + (1-\beta)(2-\beta)L_3'(p) - 2(1-\beta)(1-p)L_3''(p) \geq 0.$$

Divide both sides by $(1-p)^2$. The left-hand side of the inequality can be rewritten as

$$L_3'''(p)+\left(\frac{1-\beta}{1-p}\right)^2 L_3'(p)-2\frac{1-\beta}{1-p}L_3''(p)+\frac{1-\beta}{(1-p)^2}L_3'(p),$$

which is non-negative for any $\beta\in(0,1]$, since the sum of the first three terms is non-negative by (16).

Consider $L_8(p)$ and prove

(17) $$\lambda^2 L_4'(p)+L_4'''(p)-2\lambda L_4''(p)\geq 0.$$

Denote the left-hand side as $v(p)$. Then, by (7),

$$\frac{v(p)}{\beta_2 L_{\lambda_2}'(p)\left(1-L_{\lambda_2}(p)\right)^{\beta_2-3}}=\left\{\left[(1-\beta_2)L_{\lambda_2}'(p)+\lambda_2\left(1-L_{\lambda_2}(p)\right)\right]-\lambda\left(1-L_{\lambda_2}(p)\right)\right\}^2$$
$$+(1-\beta_2)\left[L_{\lambda_2}'(p)+\lambda_2\left(1-L_{\lambda_2}(p)\right)\right]L_{\lambda_2}'(p)\geq 0$$

holds, which implies that (17) is true for any $\lambda\neq 0$.

Consider $L_9(p)$ and prove

$$(1-\beta)(2-\beta)L_4'(p)+(1-p)^2 L_4'''(p)-2(1-\beta)(1-p)L_4''(p)\geq 0.$$

Divide both sides of this inequality by $(1-p)^2$ and then the right-hand side can be rewritten as

$$\left(\frac{1-\beta}{1-p}\right)^2 L_4'(p)+L_4'''(p)-2\frac{1-\beta}{1-p}L_4''(p)+\frac{1-\beta}{(1-p)^2}L_4'(p),$$

which is non-negative because the sum of the first three terms is non-negative by (17), if $\beta\in(0,1]$.

Finally, consider $L_{10}(p)$. We have to prove

$$w(p)=L_3'''(p)L_4'(p)+L_3'(p)L_4'''(p)-2L_3''(p)L_4''(p)\geq 0.$$

We can verify that

$$\frac{w(p)}{\beta_1\beta_2 L_{\lambda_2}'(p)L_{\lambda_1}'(1-p)L_{\lambda_1}(1-p)^{\beta_1-3}\left(1-L_{\lambda_2}(p)\right)^{\beta_2-3}}$$
$$=\left\{\left(1-L_{\lambda_2}(p)\right)\left[(1-\beta_1)L_{\lambda_1}'(1-p)-\lambda_1 L_{\lambda_1}(1-p)\right]-L_{\lambda_1}(1-p)\left[(1-\beta_2)L_{\lambda_2}'(p)+\lambda_2\left(1-L_{\lambda_2}(p)\right)\right]\right\}^2$$
$$+(1-\beta_1)L_{\lambda_1}'(1-p)\left[L_{\lambda_1}'(1-p)-\lambda_1 L_{\lambda_1}(1-p)\right]\left(1-L_{\lambda_2}(p)\right)^2$$
$$+(1-\beta_2)L_{\lambda_2}'(p)\left[L_{\lambda_2}'(p)+\lambda_2\left(1-L_{\lambda_2}(p)\right)\right]L_{\lambda_1}(1-p)^2\geq 0$$

is true by (6) and (7). We have thus obtained the desired result.

3.3. Lorenz model examples

While there are only five GP members, we have thirty-one elements in X. Therefore, millions of weighted product Lorenz models can be created by using the elements of X inclusively and the third statement of Theorem 1, even if we refrain from using an element of X repeatedly in building a model. Clearly, the method can be reinforced by adding even a single new member to the GP group. Then Theorem 2 implies that X will contain sixty-three elements and thus give rise to an immense expansion in the availability of the weighted product models.

Nevertheless, we can use less desirable models. For example, $L(p) = pA^{p-1}$, where $A > 0$, is a Lorenz curve with $L'''(p) \geq 0$, which is suggested by Gupta (1984).[§] WNS (2007) provide another option

$$H(p) = 1 - e^{-\gamma p}(1-p)^{\beta}, \quad \beta \in (0,1], \quad 0 \leq \gamma + \beta \leq \sqrt{\beta}$$

with $H'''(p) \geq 0$. Adding even a single such model to the GP group, we can also create about 2^{63} weighted product models by the second statement of Theorem 1. However, the admissible range of the exponential parameter α_i for the component with, say, $H(p)$, involved is no longer $\alpha_i \geq 0$. For example,

$$\text{(18)} \qquad p^{\alpha}\left\{\delta\left[1 - e^{-\gamma p}(1-p)^{\beta}\right] + (1-\delta)\left[1 - L_{\lambda_1}(1-p)^{\beta_1}\right]\right\}^{\eta}$$

is a Lorenz curve, where $\eta \geq 1/2$ must be imposed by Theorem 1. Other parameter ranges for (18) are $\alpha \geq 0$, $\alpha + \eta \geq 1$, $\beta \in (0,1]$, $0 \leq \gamma + \beta \leq \sqrt{\beta}$, $\delta \in [0,1]$, $\beta_1 \in (0,1]$ and $\lambda_1 \in (-\infty, 0) \cup (0, \ln \beta_1^{-1}]$.

We can expect that over-parameterization will occur in general, for example, when we use too many elements of X as components in a model. There are three guiding lessons in creating the weighted product models. One is that we should include different models of X rather than use some of them repeatedly in a single model created, as done for the model specified in (14). The second is that convex combination components are better choices. The third is that the components with

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[§] Letting $A = e^{\alpha}$ with $\alpha \geq 0$ in Gupta's (1984) model, we obtain an equivalent model $pe^{\alpha(p-1)}$, which is also given in Kakwani & Podder (1973).

$L_3(p)$, $L_4(p)$ or $H(p)$ involved are more satisfactory in constructing the models.

Therefore, Lorenz models abound by Theorem 1 and 3. The following are a few simple examples with only GP members involved.

$$p^{\alpha}\left\{\delta L_{\lambda}(p)+(1-\delta)\left[1-\left(1-L_{\lambda_2}(p)\right)^{\beta_2}\right]\right\}^{\eta}, \tag{19}$$

$$p^{\alpha}\left\{\delta_1\left(1-(1-p)^{\beta}\right)+\delta_2 L_{\lambda}(p)+\delta_3\left(1-L_{\lambda_1}(1-p)^{\beta_1}\right)\right\}^{\eta}, \tag{20}$$

$$\left(1-L_{\lambda_1}(1-p)^{\beta_1}\right)^{\alpha}\left\{\delta p+(1-\delta)\left[1-\left(1-L_{\lambda_2}(p)\right)^{\beta_2}\right]\right\}^{\eta}, \tag{21}$$

$$\left[\delta p+(1-\delta)L_{\lambda}(p)\right]^{\alpha}\left\{\delta_1\left(1-L_{\lambda_1}(1-p)^{\beta_1}\right)+(1-\delta_1)L_{\lambda_0}(p)\right\}^{\eta}, \tag{22}$$

$$p^{\alpha}\left[\delta p+(1-\delta)L_{\lambda}(p)\right]^{\alpha_1}\left(1-(1-p)^{\beta}\right)^{\eta}. \tag{23}$$

The parameter ranges are at their maximum for all these models. Namely, $\alpha \geq 0$, $\eta \geq 0$, $\alpha+\eta \geq 1$, $\alpha_1 \geq 0$, $\lambda_0 > 0$, $\delta, \delta_1, \delta_2, \delta_3 \in [0,1]$ and $\delta_3 = 1-\delta_1-\delta_2$. Those not mentioned are defined in (2)-(5). The parameter ranges above seem to be complicated. Nevertheless, they are very convenient to enforce by analogous parameter transformations to those used in WNS (2007). For example, the condition for the three weight coefficients in (20) can be enforced by parameter transformations

$$\delta_1 = \sin^2\theta_1, \quad \delta_2 = \cos^2\theta_1 \sin^2\theta_2, \quad \delta_3 = \cos^2\theta_1 \cos^2\theta_2 = 1-\delta_1-\delta_2,$$

where θ_1 and θ_2 are two new parameter variables.

4. Empirical calculations

We present two sets of examples in this section to show the efficiency of our models. One uses actual data and the other simulated data. The actual data example uses US income distribution data from Basmann *et al*. (1993). Following this, we provide estimated results of a dataset taken from an artificially created Lorenz curve.

4.1 US income data estimation

We use US income distribution data from Basmann *et al*. (1993) in our empirical calculations. There are seven years of data for 1977-1983 and there are 99 points on the empirical Lorenz curve for each year; that is, $\left(p_i, L(p_i)\right)_{i=1}^{99}$ with $p_i = 0.01i$, where $L(p)$ denotes the empirical Lorenz curve. We only consider using models given in (13) and (18)-(23) respectively to fit the 99 points.

We only display the results for the data for 1977 in Table 1 and for 1983 in Table 3, since Basmann *et al*. (1993) offered observed data with more accuracy for these two years (see Basmann *et al*. 1993, Table 3.11-12) than for other years. To conserve space, we select only about 40 points both on the empirical and estimated Lorenz curves, $L(p)$ and $\hat{L}(p)$, to display in Tables 1 and 3. On the row next to the last, the largest absolute error MAXABS=$\max_{1\le i\le 99} |\hat{L}(p_i) - L(p_i)|$ is given. We have also used two other mean error measures, denoted as MSE and MAE respectively, to facilitate comparison of the models, in which MSE $= n^{-1}\sum_{i=1}^{n}\left(\hat{L}(p_i) - L(p_i)\right)^2$, MAE $= n^{-1}\sum_{i=1}^{n}\left|\hat{L}(p_i) - L(p_i)\right|$ and $n = 99$. Similarly, Sarabia *et al*. (1999, 2001) also used these three measures in the development of their Lorenz models.

From the MAXABS measures in Table 1, model (13) and (23) are inferior to the others, while model (22) performs the best. The MAXABS value is only about 0.023%, while the MSE measure is only 0.0067×10^{-6}. Considering that we are fitting an empirical Lorenz curve with as many as 99 points, this level of error is very small. The other four models are not distinguishable by MAE. Apart from models (13) and (23), each model is a good global approximation of the data. Their MAXABS measures are not larger than 0.073%, implying that the error of the estimated Lorenz curve begins to occur at most at the fourth digit after the decimal point.

For the estimation results for 1983 in Table 3, the MAXABS measures show that the fitted results for model (20) and (22) are better than the others. They offer good global approximations to the empirical Lorenz curve, while (22) seems slightly better than (20) because there are two error measures out of three which support this model. Model (13) and (23) again perform slightly poorer than the others. Our estimated Gini indices, listed in the last row of Tables 1 and 3 for 1977 and 1983 are only slightly larger than the actual Ginis provided by Choeng (2002), which are 0.3682 for 1977 and 0.3896 for 1983 respectively. These actual Ginis can be understood as the lower limits of the empirical Gini indices, since they are calculated from the Lorenz curve obtained as the piecewise linear interpolation over the 99 data points.

Table 2 and 4 give parameter estimations, making it is easy to verify that the parameter constraints have been enforced, which implies that each fitted curve satisfies the definition of the Lorenz curve. The Levenberg-Marquardt algorithm (Dennis & Schnabel, 1983) with finite-deference Jacobian is employed to solve the non-linear least squares (NLS) problem so as to obtain the estimations. This allows us to avoid generating the routines of the analytical Jacobian matrixes in which we are apt to make errors, especially when we deal with a model with many parameters.

We observe that the performance of model (23) is much poorer, if $L_\lambda(p)$ is replaced with the model specified in (2) in the second component of (23), so as to result in a model which nests (14). This implies that (14) does not satisfactorily cope with the data configuration here. Therefore, we can conclude that there may be many models created from X which are superior to some traditional models in the literature.

Estimation results for the US data for 1977-1983 can be found in Basmann *et al.* (1993). The same authors also provide results for the US 1977 data in Basmann *et al.* (1990). Other estimation results for the US data for 1977 can be found in Ryn & Slottje (1996), Sarabia *et al.* (1999) and WNS (2007). Our estimation results for US data for 1978-1982, which are not given here to conserve space, paint the same picture as depicted in this paper; ie. models (18)-(22) are satisfactory for the data for almost all the years and these are superior to model (13) and (23).

4.2 An artificial example

We create a special dataset to test our models. The data can be very hard to fit, since the underlying Lorenz curve contains some singularity. Take $B \in (0,1)$, $M_1 > 1$, $M_2 > 1$, $h_1 = B/M_1$ and $h_2 = (1-B)/M_2$. Form the dataset $\{(p_i, L_i)\}_{i=1}^{M_1+M_2}$ with $p_i = h_1 \times i$ for $i = 1,2,\cdots,M_1$ and $p_{M_1+i} = B + h_2 \times i$ for $i = 1,2,\cdots,M_2$, where $L_i = \tilde{L}(p_i)$ is taken from the Lorenz curve

$$\tilde{L}(p) = \begin{cases} g(p) & p \in [0, B] \\ h(p) + [1 - h(1)]\left(\dfrac{p-B}{1-B}\right)^3 & p \in (B, 1] \end{cases}$$

where $g(p)$ is a non-negative and convex function on [0,1] with a second continuous derivative, satisfies $g(1) < 1$ and

$$h(p) = g(B) + g'(B)(p - B) + \frac{1}{2} g''(B)(p - B)^2 .$$

Therefore, $\tilde{L}(p)$ is a function with a second continuous derivative. But its third derivative is not continuous at $p = B$. For example, if $g(p) = (p/m)^2$ is taken, $m = 2$ and $B = 0.9$ together mean that the 90% of residents at the bottom of the distribution share only 20.25% of the total income and those 10% of residents at the top of the distribution obtain 79.75% of the total. Furthermore, the Gini index of $\tilde{L}(p)$ is as high as

$$\text{Gini} = 1 - 2\int_0^1 \tilde{L}(p)\mathrm{d}p = 1 - \left[\frac{1}{3m^2} + \left(1 - \frac{1}{m^2}\right)\frac{1-B}{4} \right] \approx 0.79583$$

Because of $L(0) = 0, L(1) = 1, L'(p) \geq 0, L''(p) \geq 0$, $\tilde{L}(p)$ is indeed a Lorenz curve.

We present results by choosing $g(p) = (p/m)^2$, $M_1 = 45$, $M_2 = 5$, $B = 0.9$, $m = 2$. The results are not as encouraging as the results with the US income data, but they do show that our models are superior to the traditional ones in the literature. This can be seen in Figures 1 to 4. They show that models (18) and (22) are very flexible, but models (13) and (23) are not satisfactory. The graphs show that the curves produced by (13) and (23) cannot bend into an angle, which is very close to the coordinate point $(0,1)$, as other models do. This may be the most significant weakness of the traditional models, since both (13) and (23) are extensions of some traditional models. Tables 5 and 6 give the estimated results. From Table 5 we can see that model (18) is the best for the data in terms of the three measures. The Gini index estimation is also the closest out of all the models using actual values. Model (19) seems to be the second best while (22) is only a modest model, which was the best for US data as we have shown above.

5. Conclusion

We have presented a general method to create Lorenz models of the weighted product form. The method rests on finding a set of parametric Lorenz models. We find that an ideal situation is when, for each element of the set, the ratio of its second derivative to its first derivative is increasing. We have presented a set X of Lorenz models which possess this property. Hence, we can create a large number of parametric Lorenz models. Moreover, our results provide evidence that we can have models with good global approximation to the actual data. Thus, we conclude that, while the models in X are limited, our method can be useful in practice.

The most significant feature of our method is that we can increase the power of the method by increasing the set X found. This could be an interesting topic for further research. Another further research subject could be finding methods to determine the most favorable model/models among the ones created from X. It could be possible to find completely new set/sets, with elements possessing the same property as the models in X, so as to obtain other sets of weighted product Lorenz models.

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Table 1. Actual and estimated Lorenz values for US 1977 income data

	Actual Lorenz	Estimated Lorenz						
p	$L(p)$	(13)	(18)	(19)	(20)	(21)	(22)	(23)
0.01	0.00039	0.00046	0.00057	0.00057	0.00055	0.00055	0.00056	0.00045
0.02	0.00136	0.00135	0.00159	0.00159	0.00156	0.00155	0.00158	0.00133
0.03	0.00267	0.00256	0.00291	0.00291	0.00287	0.00285	0.00290	0.00253
0.04	0.00428	0.00403	0.00448	0.00447	0.00442	0.00440	0.00447	0.00399
0.05	0.00612	0.00572	0.00626	0.00625	0.00620	0.00617	0.00625	0.00568
0.06	0.00814	0.00763	0.00823	0.00823	0.00817	0.00814	0.00824	0.00758
0.07	0.01034	0.00974	0.01039	0.01040	0.01033	0.01030	0.01040	0.00969
0.08	0.01272	0.01204	0.01273	0.01274	0.01267	0.01263	0.01274	0.01199
0.09	0.01526	0.01451	0.01523	0.01524	0.01518	0.01514	0.01525	0.01446
0.10	0.01797	0.01716	0.01789	0.01791	0.01784	0.01781	0.01792	0.01712
0.15	0.03365	0.03284	0.03350	0.03353	0.03348	0.03345	0.03353	0.03281
0.20	0.05279	0.05228	0.05272	0.05275	0.05273	0.05271	0.05273	0.05228
0.25	0.07544	0.07528	0.07542	0.07544	0.07545	0.07545	0.07541	0.07530
0.30	0.10150	0.10175	0.10160	0.10159	0.10163	0.10164	0.10156	0.10178
0.35	0.13114	0.13167	0.13128	0.13125	0.13129	0.13132	0.13122	0.13171
0.40	0.16440	0.16508	0.16454	0.16449	0.16454	0.16457	0.16450	0.16512
0.45	0.20143	0.20207	0.20150	0.20146	0.20149	0.20151	0.20149	0.20210
0.50	0.24238	0.24277	0.24231	0.24228	0.24229	0.24229	0.24234	0.24279
0.55	0.28734	0.28737	0.28713	0.28713	0.28711	0.28710	0.28721	0.28737
0.60	0.33637	0.33611	0.33619	0.33623	0.33618	0.33616	0.33629	0.33610
0.65	0.38977	0.38932	0.38974	0.38981	0.38975	0.38972	0.38981	0.38931
0.70	0.44808	0.44745	0.44813	0.44819	0.44815	0.44813	0.44810	0.44742
0.75	0.51162	0.51109	0.51182	0.51184	0.51184	0.51184	0.51166	0.51105
0.80	0.58141	0.58111	0.58154	0.58149	0.58154	0.58156	0.58137	0.58107
0.85	0.65870	0.65887	0.65859	0.65849	0.65855	0.65859	0.65866	0.65883
0.90	0.74588	0.74671	0.74565	0.74563	0.74563	0.74564	0.74593	0.74671
0.91	0.76491	0.76585	0.76471	0.76473	0.76471	0.76470	0.76495	0.76586
0.92	0.78462	0.78564	0.78450	0.78455	0.78451	0.78449	0.78464	0.78567
0.93	0.80517	0.80616	0.80510	0.80519	0.80513	0.80510	0.80511	0.80621
0.94	0.82659	0.82753	0.82667	0.82679	0.82671	0.82668	0.82652	0.82759
0.95	0.84908	0.84989	0.84938	0.84951	0.84943	0.84939	0.84908	0.84998
0.96	0.87307	0.87346	0.87348	0.87359	0.87352	0.87349	0.87309	0.87357
0.97	0.89890	0.89858	0.89931	0.89937	0.89934	0.89932	0.89900	0.89872
0.98	0.92759	0.92587	0.92745	0.92741	0.92744	0.92746	0.92746	0.92604
0.99	0.95956	0.95673	0.95904	0.95883	0.95897	0.95904	0.95960	0.95692
MSE$\times 10^6$		0.38349	0.02463	0.03077	0.02906	0.02979	0.00670	0.39291
MAE		0.00050	0.00013	0.00013	0.00014	0.00015	0.00007	0.00052
MAXABS		0.00283	0.00052	0.00073	0.00059	0.00052	0.00023	0.00264
Gini		0.36846	0.36828	0.36828	0.36829	0.36829	0.36827	0.36845

Note: The column below (13), for example, contains the estimated Lorenz values of the model specified in Equation (13).

Table 2. Parameter Estimation for US 1977 income data.

Param.	(13)	(18)	(19)	(20)	(21)	(22)	(23)
α	0.235334	0.897784	0.000000	0.000000	0.874604	1.469687	0.037016
γ	--	-0.481669	--	--	--	--	--
β	--	0.805361	--	0.777232	--	--	0.780748
λ	--	--	0.733345	0.766468	--	31.519441	10.000000
β_1	0.772311	0.833797	--	0.763113	0.937211	0.642972	--
λ_1	-0.224369	-20.058431	--	-16.262556	-1.063209	-45.438297	--
δ	--	0.920488	0.936439	--	0.887338	0.968906	0.000377
δ_1	--	--	--	0.146741	--	0.931660	--
δ_2	--	--	--	0.801288	--	--	--
λ_0	--	--	--	--	--	29.019418	--
α_1	0.312752	--	--	--	--	--	0.011431
β_2	0.772310	--	0.733406	--	0.746970	--	--
λ_2	0.224379	--	14.844698	--	16.914844	--	--
η	1.015054	0.574204	1.478596	1.488960	0.619459	0.020492	1.527898

Note: The column below (13), for example, contains the parameter estimates of the model specified in Equation (13).

Table 3. Actual and estimated Lorenz values for US 1983 income data

	Actual Lorenz	Estimated Lorenz						
p	$L(p)$	(13)	(18)	(19)	(20)	(21)	(22)	(23)
0.01	0.00026	0.00039	0.00042	0.00041	0.00038	0.00041	0.00040	0.00036
0.02	0.00098	0.00117	0.00124	0.00121	0.00115	0.00120	0.00119	0.00110
0.03	0.00202	0.00223	0.00233	0.00230	0.00220	0.00228	0.00226	0.00212
0.04	0.00332	0.00352	0.00366	0.00361	0.00349	0.00359	0.00356	0.00339
0.05	0.00482	0.00503	0.00520	0.00514	0.00499	0.00511	0.00508	0.00487
0.06	0.00658	0.00674	0.00693	0.00686	0.00670	0.00683	0.00679	0.00655
0.07	0.00849	0.00864	0.00884	0.00877	0.00859	0.00873	0.00869	0.00842
0.08	0.01059	0.01071	0.01092	0.01085	0.01066	0.01080	0.01076	0.01048
0.09	0.01284	0.01294	0.01316	0.01309	0.01290	0.01304	0.01300	0.01270
0.10	0.01527	0.01535	0.01557	0.01549	0.01530	0.01544	0.01541	0.01510
0.15	0.02974	0.02967	0.02987	0.02980	0.02963	0.02973	0.02971	0.02943
0.20	0.04775	0.04764	0.04774	0.04770	0.04761	0.04762	0.04763	0.04745
0.25	0.06909	0.06907	0.06906	0.06905	0.06905	0.06898	0.06901	0.06897
0.30	0.09389	0.09393	0.09381	0.09383	0.09391	0.09377	0.09383	0.09392
0.35	0.12209	0.12223	0.12203	0.12208	0.12220	0.12204	0.12211	0.12231
0.40	0.15388	0.15405	0.15382	0.15387	0.15401	0.15386	0.15394	0.15418
0.45	0.18941	0.18952	0.18930	0.18936	0.18947	0.18937	0.18945	0.18967
0.50	0.22881	0.22880	0.22867	0.22870	0.22876	0.22874	0.22879	0.22894
0.55	0.27228	0.27213	0.27213	0.27213	0.27212	0.27219	0.27220	0.27223
0.60	0.31995	0.31981	0.31996	0.31994	0.31985	0.32000	0.31996	0.31984
0.65	0.37237	0.37221	0.37253	0.37248	0.37234	0.37254	0.37242	0.37218
0.70	0.43002	0.42986	0.43028	0.43021	0.43007	0.43025	0.43008	0.42977
0.75	0.49366	0.49345	0.49382	0.49378	0.49371	0.49377	0.49361	0.49333
0.80	0.56409	0.56398	0.56408	0.56408	0.56414	0.56403	0.56400	0.56387
0.85	0.64273	0.64300	0.64256	0.64262	0.64280	0.64255	0.64278	0.64295
0.90	0.73252	0.73318	0.73226	0.73232	0.73239	0.73231	0.73257	0.73323
0.91	0.75223	0.75297	0.75203	0.75209	0.75210	0.75210	0.75228	0.75304
0.92	0.77270	0.77348	0.77261	0.77264	0.77258	0.77267	0.77273	0.77357
0.93	0.79407	0.79481	0.79409	0.79409	0.79395	0.79414	0.79404	0.79492
0.94	0.81636	0.81709	0.81662	0.81660	0.81640	0.81666	0.81639	0.81721
0.95	0.83991	0.84048	0.84042	0.84036	0.84012	0.84041	0.84000	0.84060
0.96	0.86597	0.86522	0.86573	0.86565	0.86544	0.86566	0.86522	0.86534
0.97	0.89225	0.89171	0.89295	0.89287	0.89277	0.89280	0.89255	0.89180
0.98	0.92235	0.92062	0.92269	0.92265	0.92282	0.92243	0.92278	0.92067
0.99	0.95742	0.95350	0.95621	0.95625	0.95681	0.95586	0.95716	0.95349
MSE$\times10^6$		0.25027	0.05147	0.03805	0.02178	0.05003	0.01729	0.27795
MAE		0.00024	0.00016	0.00013	0.00010	0.00014	0.00009	0.00029
MAXABS		0.00392	0.00121	0.00117	0.00061	0.00156	0.00075	0.00393
Gini		0.38972	0.38962	0.38963	0.38966	0.38968	0.38963	0.38978

Note: The column entitled (13), for example, contains the estimated Lorenz values of the model specified in Equation (13).

Table 4. Parameter Estimation for US 1983 income data.

Param.	(13)	(18)	(19)	(20)	(21)	(22)	(23)
α	0.000000	0.351621	0.000000	0.000000	1.200618	1.403729	0.000000
γ	--	-0.204183	--	--	--	--	--
β	--	0.807252	--	0.871838	--	--	0.769854
λ	--	--	23.170670	28.797345	--	8.660097	3.835978
β_1	0.791390	1.000000	--	0.843230	0.802114	0.738203	--
λ_1	-0.629466	-21.464301	--	-1.484770	-0.500439	-3.915562	--
δ	--	0.975020	0.017210	--	0.925510	0.950910	0.182648
δ_1	--	--	--	0.702104	--	0.854684	--
δ_2	--	--	--	0.018992	--	--	--
λ_0	--	--	--	--	--	48.396028	--
α_1	0.185379	--	--	--	--	--	0.062422
β_2	0.780674	--	0.832052	--	0.999973	--	--
λ_2	0.158011	--	0.368162	--	20.263162	--	--
η	1.398505	1.198917	1.562071	1.594477	0.364908	0.170979	1.554361

Note: The column entitled (13), for example, contains the parameter estimates of the model specified in Equation (13)

Table 5. Actual and estimated Lorenz values for the artificial example

	Actual Lorenz	Estimated Lorenz						
p	$L(p)$	(13)	(18)	(19)	(20)	(21)	(22)	(23)
0.02	0.00010	0.00040	0.00020	0.00014	0.00027	0.00114	0.00022	0.00012
0.04	0.00040	0.00100	0.00068	0.00051	0.00086	0.00250	0.00073	0.00041
0.06	0.00090	0.00172	0.00139	0.00110	0.00169	0.00398	0.00149	0.00085
0.08	0.00160	0.00255	0.00231	0.00189	0.00273	0.00557	0.00247	0.00142
0.10	0.00250	0.00348	0.00344	0.00288	0.00397	0.00726	0.00367	0.00213
0.12	0.00360	0.00450	0.00475	0.00406	0.00540	0.00904	0.00505	0.00297
0.14	0.00490	0.00562	0.00625	0.00543	0.00700	0.01092	0.00663	0.00393
0.16	0.00640	0.00683	0.00794	0.00699	0.00878	0.01290	0.00839	0.00503
0.18	0.00810	0.00815	0.00980	0.00873	0.01072	0.01497	0.01033	0.00626
0.20	0.01000	0.00957	0.01184	0.01066	0.01282	0.01714	0.01245	0.00763
0.22	0.01210	0.01110	0.01406	0.01276	0.01508	0.01941	0.01473	0.00912
0.24	0.01440	0.01274	0.01645	0.01504	0.01750	0.02179	0.01718	0.01076
0.26	0.01690	0.01450	0.01902	0.01751	0.02008	0.02428	0.01979	0.01254
0.28	0.01960	0.01638	0.02176	0.02014	0.02281	0.02687	0.02256	0.01446
0.30	0.02250	0.01840	0.02467	0.02296	0.02569	0.02959	0.02550	0.01653
0.32	0.02560	0.02055	0.02776	0.02595	0.02873	0.03242	0.02859	0.01876
0.34	0.02890	0.02285	0.03102	0.02912	0.03193	0.03537	0.03184	0.02114
0.36	0.03240	0.02531	0.03445	0.03246	0.03527	0.03846	0.03524	0.02369
0.38	0.03610	0.02794	0.03805	0.03597	0.03878	0.04167	0.03881	0.02642
0.40	0.04000	0.03074	0.04182	0.03967	0.04244	0.04503	0.04252	0.02933
0.42	0.04410	0.03374	0.04577	0.04354	0.04626	0.04853	0.04640	0.03242
0.44	0.04840	0.03694	0.04990	0.04758	0.05025	0.05219	0.05043	0.03572
0.46	0.05290	0.04036	0.05420	0.05181	0.05439	0.05600	0.05461	0.03923
0.48	0.05760	0.04402	0.05867	0.05621	0.05870	0.05997	0.05896	0.04296
0.50	0.06250	0.04793	0.06332	0.06079	0.06319	0.06411	0.06346	0.04693
0.52	0.06760	0.05212	0.06815	0.06556	0.06784	0.06843	0.06812	0.05115
0.54	0.07290	0.05661	0.07315	0.07051	0.07267	0.07294	0.07295	0.05566
0.56	0.07840	0.06143	0.07834	0.07564	0.07768	0.07764	0.07794	0.06046
0.58	0.08410	0.06661	0.08370	0.08096	0.08288	0.08255	0.08311	0.06558
0.60	0.09000	0.07219	0.08925	0.08648	0.08827	0.08767	0.08845	0.07106
0.62	0.09610	0.07819	0.09498	0.09219	0.09386	0.09301	0.09396	0.07694
0.64	0.10240	0.08468	0.10089	0.09810	0.09966	0.09858	0.09966	0.08326
0.66	0.10890	0.09169	0.10699	0.10421	0.10568	0.10439	0.10556	0.09006
0.68	0.11560	0.09930	0.11327	0.11054	0.11193	0.11046	0.11165	0.09742
0.70	0.12250	0.10757	0.11974	0.11707	0.11841	0.11680	0.11796	0.10542
0.72	0.12960	0.11659	0.12640	0.12384	0.12515	0.12342	0.12450	0.11415
0.74	0.13690	0.12647	0.13325	0.13084	0.13217	0.13034	0.13128	0.12373
0.76	0.14440	0.13734	0.14030	0.13808	0.13948	0.13757	0.13834	0.13430
0.78	0.15210	0.14935	0.14755	0.14560	0.14712	0.14514	0.14572	0.14606
0.80	0.16000	0.16270	0.15503	0.15341	0.15514	0.15309	0.15348	0.15925
0.82	0.16810	0.17765	0.16281	0.16160	0.16363	0.16148	0.16173	0.17417
0.84	0.17640	0.19454	0.17103	0.17032	0.17279	0.17045	0.17069	0.19122
0.86	0.18490	0.21384	0.18010	0.18001	0.18303	0.18038	0.18085	0.21095
0.88	0.19360	0.23622	0.19110	0.19172	0.19539	0.19224	0.19330	0.23410
0.90	0.20250	0.26266	0.20671	0.20827	0.21233	0.20858	0.21072	0.26176
0.92	0.21760	0.29477	0.23355	0.23654	0.23977	0.23595	0.23942	0.29561
0.94	0.26890	0.33541	0.28632	0.29177	0.29105	0.28925	0.29305	0.33846
0.96	0.39240	0.39059	0.39362	0.40236	0.39358	0.39725	0.39724	0.39598
0.98	0.62410	0.47771	0.60448	0.61398	0.59800	0.60744	0.59531	0.48369
MSE $\times 10^2$		0.08731	0.00249	0.00323	0.00430	0.00466	0.00495	0.08580
MAE		0.01576	0.00297	0.00345	0.00388	0.00560	0.00416	0.01647
MAXABS		0.14639	0.01962	0.02287	0.02610	0.02035	0.02879	0.14041
Gini		0.80714	0.79574	0.79756	0.79496	0.79271	0.79604	0.80975

Note: The column entitled (13), for example, contains the estimated Lorenz values of the model specified in Equation (13).

Table 6. Parameter Estimation for the artificial example.

Param.	(13)	(18)	(19)	(20)	(21)	(22)	(23)
α	0.000011	1.178435	1.375003	1.047044	0.603505	1.317341	0.195349
γ	--	-0.342687	--	--	--	--	--
β	--	0.994803	--	0.433690	--	--	0.204987
λ	--	--	49.995350	47.308814	--	22.999078	9.981128
β_1	0.261012	0.999967	--	0.999738	0.885679	0.105896	--
λ	-1.180825	-47.936088	--	-48.300198	-2.856807	2.232953	--
δ	--	0.082380	0.923885	--	0.069032	0.875896	0.463432
δ_1	--	--	--	0.140077	--	0.168624	--
δ_2	--	--	--	0.854966	--	--	--
λ_0	--	--	--	--	--	49.999511	--
α_1	0.065356	--	--	--	--	--	0.441208
β_2	0.251353	--	0.497623	--	0.999994	--	--
λ_2	1.518205	--	-0.695758	--	49.996849	--	--
η	1.212493	0.571091	0.511652	0.614332	0.501820	0.443796	1.130556

Note: The column entitled (13), for example, contains the parameter estimates of the model specified in Equation (13)

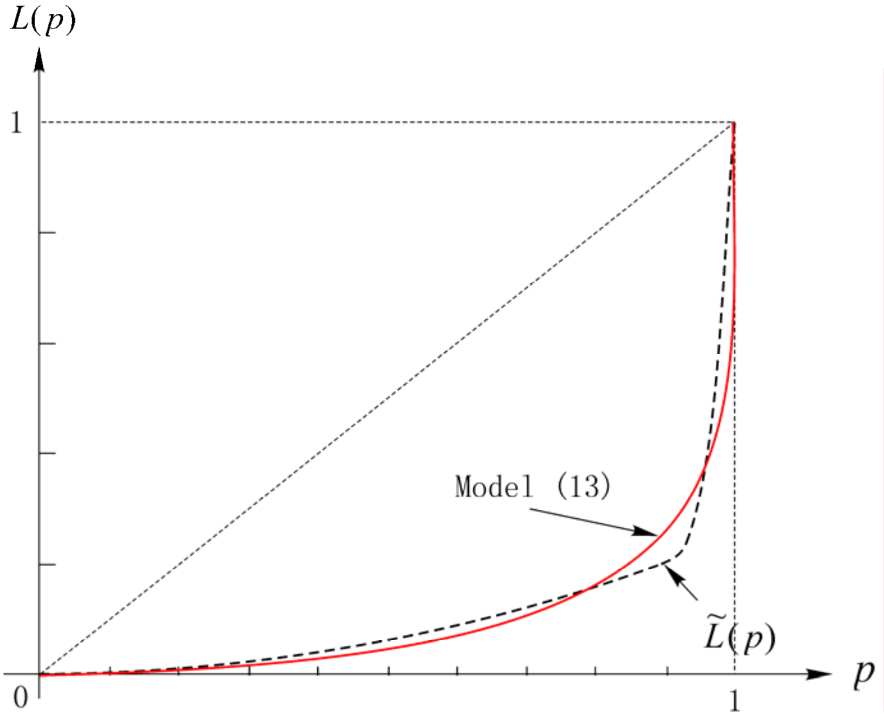

Fig. 1 Estimation of model (13)

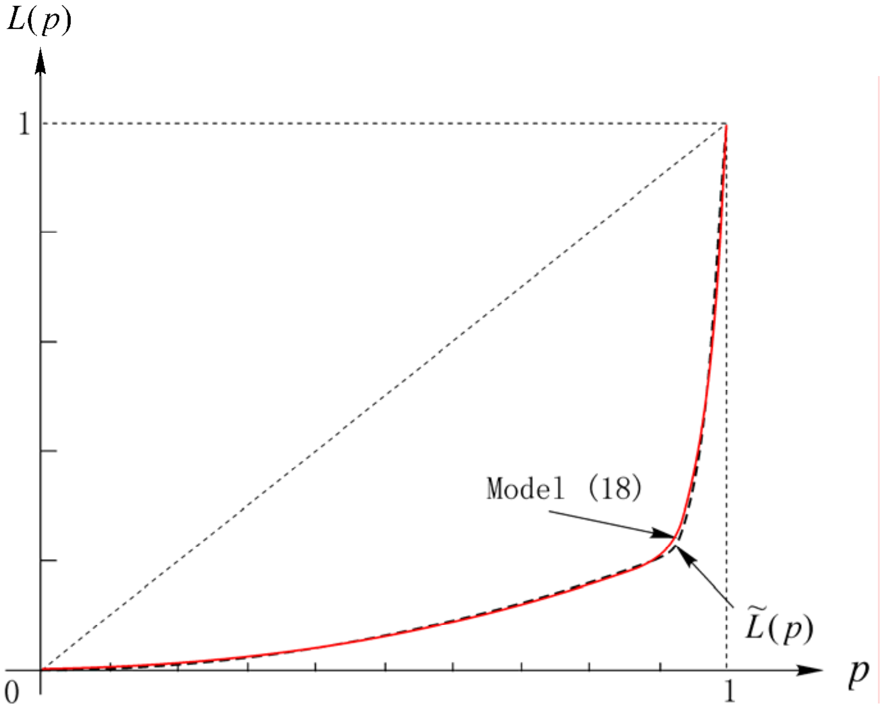

Fig. 2 Estimation of model (18)

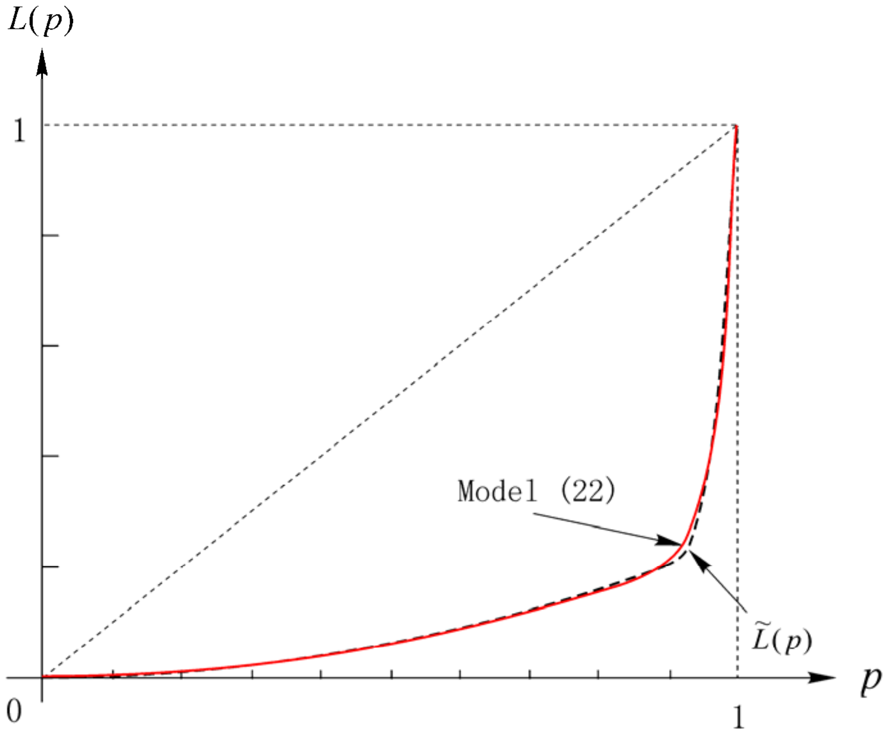

Fig. 3 Estimation of model (22)

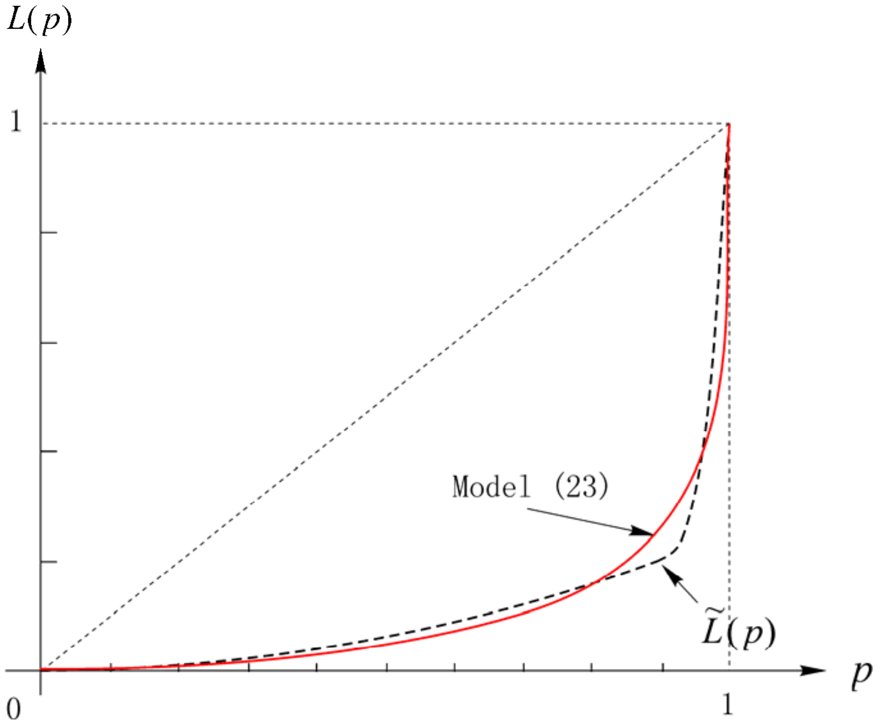

Fig. 4 Estimation of model (23)